Weakly Mixing Tiling Flows Arising from Interval Exchange Transformations

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A Dissertation submitted to

The Faculty of

Columbian School of Arts and Sciences
of The George Washington University in partial satisfaction
of the requirements for the degree of Doctor of Philosophy

May 17, 1998

Dissertation directed by

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Acknowledgments

I would like to acknowledge God for all my blessings and my wife for her support during my pursuit of my degree. I also acknowledge my advisor Dr. E. A. Robinson, Jr.. His time and guidance was invaluable to my studies. Without his commitment I would not be in a position to earn my degree.

Summary

We give an explicit example of a pseudo-Anosov diffeomorphism with a non-Pisot expansion coefficient on a genus 2 surface. The flow along the nonsingular leaves of the expanding foliation is minimal and uniquely ergodic (for surface area). Constructing a Markov partition for the diffeomorphism, the sets induce tilings of $\mathbf{R}$ on the orbits of the flow. Thus, the corresponding tiling dynamical system is an almost 1:1 extension of the flow, which provides a "smooth model". The corresponding tiling dynamical system is self-similar and weakly, but not strongly, mixing for the action by translation.

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Chapter 1

Introduction

In this dissertation we give an explicit example of a pseudo-Anosov diffeomorphism D on a surface M such that the expansion coefficient θ is not a Pisot number. Briefly, a pseudo-Anosov diffeomorphism on a surface is a diffeomorphism which has uniform contraction and expansion on the stable and unstable subspaces, respectively. A Pisot number is an algebraic integer larger than 1, all of whose conjugates have modulus less than 1 [ST87]. The diffeomorphism D is constructed in Chapter 4 on a surface, which is the phase space of a suspension of an interval exchange transformation T. The corresponding expanding foliation $\mathcal{F}$ of the diffeomorphism D has one singularity with total angle 6π. This implies M is a surface of genus 2. Next, we consider the unit speed flow F^t along the nonsingular leaves of the expanding foliation. In Chapter 2, the transformation T is defined and shown to be self-induced. In Chapter 3, we prove that T is uniquely ergodic by using a substitution system.

Previously, all explicit examples of pseudo-Anosov diffeomorphisms have had expansion coefficients that are Pisot numbers. For example, P. Arnoux [Arn88] constructed a pseudo-Anosov diffeomorphism on a surface of genus 3 with Pisot expansion coefficient θ satisfying $\theta^3 - \theta^2 - \theta - 1 = 0$. He showed this example is semi-conjugate (almost 1:1) to a hyperbolic diffeomorphism on the 3-torus. In earlier work [Arn81], he constructed, for each $g \geq 2$, a pseudo-Anosov diffeomorphism on a surface of genus g whose expansion coefficient is Pisot of degree g. Also, W. Veech [Vee82] gave a general recipe for constructing pseudo-Anosov diffeomorphisms using interval exchange transformations. He paid special attention to transformations that "induce" themselves on smaller intervals. Our construction uses this induction process, which was first described by G. Rauzy [Rau79].

We construct a Markov partition $\mathcal{P}$ for D on M, in which the sets of $\mathcal{P}$ induce tilings of $\mathbf{R}$ on the orbits of F^t in Chapter 5. The corresponding tiling dynamical system is, then, an "almost 1:1 extension" of F^t, satisfying an inflation map realized by the diffeomorphism D. This provides an example of a weakly mixing tiling dynamical system, which has a "smooth model". In the last section of Chapter 5, we show how to construct a "Cartesian square" of this 1-dimensional tiling dynamical system to make a 2-dimensional tiling with a smooth model which is "non-Pisot" and, thus, weakly mixing. An example of a 2-dimensional weakly mixing tiling dynamical system determined by a substitution on two letters is presented by D. Berend and C. Radin [BR93]. For perspective, one should compare this to the dynamical system arising from Penrose tilings, which are modeled by translations on a 4-torus, studied by E. A. Robinson, Jr. [Rob96]. In particular, the Penrose tiling dynamical system has pure point spectrum and, thus, is not weakly mixing. It has a Pisot expansion constant.

In Chapter 6, we define the notion of a generalized β-transformation in terms of a self-induced interval exchange transformation. We study the natural extension of this generalized β-transformation and show it is isomorphic to the inverse of the pseudo-Anosov diffeomorphism D.

Finally, in Chapter 7 we address conclusions and hypotheses of the topics investigated in this dissertation. The work presented here leads to more questions concerning the ergodic properties of maps and numbers.

In Appendix A, Mathematica programs are given that were created to search for interval exchange transformations which induce themselves on a subinterval whose length is an inverse non-Pisot factor of the original interval. Other programs calculate the dimensions and graph the suspension surface above a transformation. In Appendix B, the actual values of the vectors and permutation studied in this dissertation are shown.

Chapter 2

Interval Exchange Transformations

2.1 Definitions

An *interval exchange transformation* $T : J \to J$ is an orientation preserving, finite piecewise isometry on an interval J. The transformation is defined on m intervals and can be characterized by a positive vector $\vec{\lambda} > 0$ in $\mathbf{R}^m$ and a permutation $\pi \in S_m$. The left-closed, right-open subintervals have lengths $\lambda_1, \lambda_2, ..., \lambda_m$, so the length of J is $\sum_{i=1}^{m} \lambda_i$. The permutation π describes the rearranging of the subintervals by the interval exchange transformation. Using this fact, we can write $T = T_{\vec{\lambda},\pi}$. Thus, T is continuous, except possibly at the endpoints of the subintervals, and right continuous everywhere. Often there is the requirement, which is not used here, that $|\vec{\lambda}| = \sum_{i=1}^{m} \lambda_i = 1$, in which case the transformations are in 1:1 correspondence with points in $\triangle_{m-1} \times S_m$ where $\triangle_{m-1} = \{\vec{\lambda} \in \mathbf{R}_+^m : |\vec{\lambda}| = 1\}$.

2.2 Rauzy Induction

We can define an "induced" interval exchange transformation $T' : J' \to J'$ where J' is a subinterval of J by considering the *first return map* on $J' \subset J$. For each $s \in J'$, let $n(s) > 0$ be the smallest positive integer such that $T^{n(s)}s \in J'$. Since T preserves Lebesgue measure, the Poincaré Recurrence Theorem [Wal81] implies that $n(s)$ is well defined for almost every $s \in J'$. The induced transformation $T' : J' \to J'$ is defined by $T's = T^{n(s)}s$. V. Oseledec proved that T' is an interval exchange transformation defined on no more that $m+2$ intervals [Sin76]. The induced transformation T' considered here is defined on the subinterval $J' = [0, r)$ where $r = |J| - \min\{\lambda_m, \lambda_{\pi^{-1}m}\}$. Using this subinterval, the induced transformation remains defined on m intervals [Rau79].

Rauzy [Rau79] proved that there exist an $m \times m$ matrix A with entries 0,1 and a map $c : S_m \to S_m$ such that $T' = T_{A^{-1}\lambda, c\pi}$ for induction on $J' = [0, r)$. For $\lambda_m < \lambda_{\pi^{-1}m}$, we say the induction is an *a-induction* where

$$A_a(\pi) = \left(\begin{array}{c|ccccc} I_{\pi^{-1}m} & & & 0 & & \\ & 1 & 0 & 0 & \dots & 0 \\ \hline & 0 & 1 & 0 & \dots & 0 \\ & . & 0 & 1 & \dots & 0 \\ 0 & . & . & & & 0 \\ & 0 & . & & & 1 \\ & 1 & 0 & 0 & \dots & 0 \end{array} \right) \tag{2.1}$$

and

$$(c_a\pi)j = \begin{cases} \pi j, & j \leq \pi^{-1}m, \\ \pi m, & j = \pi^{-1}m + 1, \\ \pi(j-1), & \text{otherwise.} \end{cases}$$

For $\lambda_m > \lambda_{\pi^{-1}m}$, we say the induction on $[0, r)$ is a *b-induction* where

$$A_b(\pi) = \left(\begin{array}{ccccccc|c} & & & & & & & 0 \\ & & & I_{m-1} & & & & . \\ & & & & & & & . \\ & & & & & & & 0 \\ \hline 0 & \dots & 0 & 1 & 0 & \dots & 0 & 1 \end{array} \right) \tag{2.2}$$

where the 1 in the last row of $A_b(\pi)$ occurs at the $\pi^{-1}m$ position and

$$(c_b\pi)j = \begin{cases} \pi j, & \pi j \leq \pi m \\ \pi j + 1, & \pi m < \pi j < m \\ \pi m + 1, & \pi j = m. \end{cases}$$

Note that if we induce again, this time on the new interval $[0, r')$, then we obtain the induced transformation $T_{A^{-1}\lambda, c^2\pi}$ where matrix $A = A_1 A_2$ is the product of the matrices defined in the two induction steps. This induction can be continued for any finite number of steps.

Graph G, shown in Figure 2.1, describes the induction on the *Rauzy class* of interval exchange transformations on four intervals. The Rauzy class consists of transformations with *irreducible permutations* ($\pi(\{1, 2, ..., j\}) \neq \{1, 2, ..., j\}$ for any $j = 1, 2, 3$) and three discontinuities. The induction action preserves the irreducible property of the permutations in the Rauzy class. Almost every interval exchange transformation in this Rauzy class corresponds to an infinite path in G. Likewise, almost every infinite path gives rise to a unique interval exchange transformation. Given a transformation $T_{\vec{\lambda},\pi}$, we start at the vertex π and repeatily induce, following an a-edge or b-edge according to the condition $\lambda_m < \lambda_{\pi^{-1}m}$ or $\lambda_m > \lambda_{\pi^{-1}m}$, respectively. Transformations that self-induce correspond to closed paths in G which are infinitely repeated. This occurs when $(A_1 A_2 ... A_k)^{-1}\vec{\lambda} = \alpha\vec{\lambda}$ and $c^k\pi = \pi$, for some $\alpha > 0$ and $k \in \mathbf{N}$.

The matrix S, defined in (2.3), is the adjacency matrix for graph G, according to the ordering $(134)(2), (124)(3), (1243), (14)(23), (142)(3), (1342), (143)(2)$ of permutations. For each $k \in \mathbf{N}$, we use matrix S to determine paths of length k from vertex i to vertex j by calculating the (i, j)-entry of S^k. Note that we must use "noncommutative multiplication" on the entries of S because following edge a and then edge b is not equivalent to following

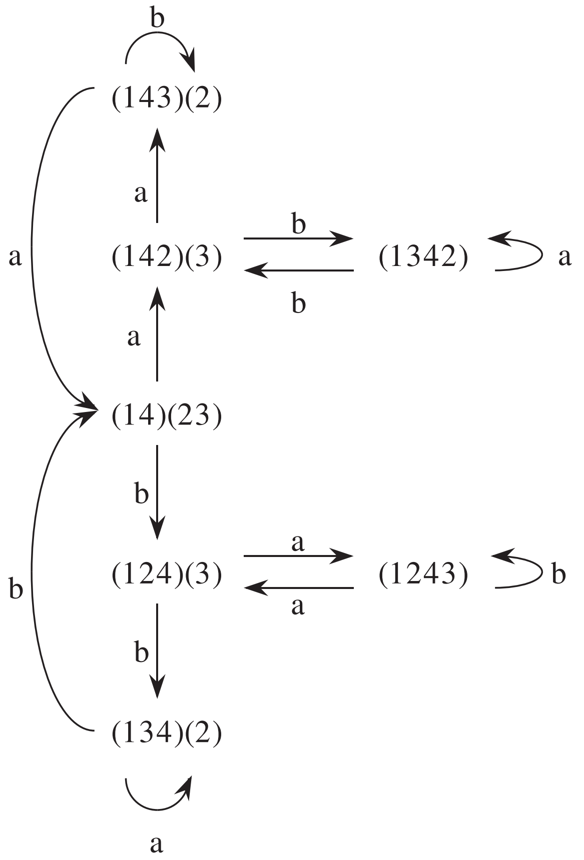

Figure 2.1: Induction graph G of Rauzy class.

edge b and then edge a.

$$S = \begin{pmatrix} a & 0 & 0 & b & 0 & 0 & 0 \\ b & 0 & a & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & a \\ 0 & 0 & 0 & 0 & b & a & 0 \\ 0 & 0 & 0 & a & 0 & 0 & b \end{pmatrix} \tag{2.3}$$

For reasons that will be apparent in Chapter 5, our goal is to find a transformation T that induces itself on a subinterval $J' = \frac{1}{\theta}J$, meaning subintervals of J' are defined by $\vec{\lambda}' = \frac{1}{\theta}\vec{\lambda}$, where θ is not a Pisot number. Recall a *Pisot number* is a real algebraic integer $\alpha > 1$, all of whose conjugates have modulus less than 1 [ST87]. This evolves finding a closed path in graph G such that $A^{-1}\vec{\lambda} = \frac{1}{\theta}\vec{\lambda}$.

Theorem 2.2.1 *There is a closed path of length 8 such that the corresponding Rauzy induction matrix has a non-Pisot, Perron-Frobenius eigenvalue. This length is the shortest possible length for a path of this type.*

Theorem 2.2.1 is proved by computations made using the Mathematica programs given in Appendix A. The explicit path is given in Section 2.4.

2.3 The Method of Veech

The construction of the pseudo-Anosov diffeomorphism presented in this dissertation is the realization of a method outlined by W. Veech [Vee82]. Consider an interval exchange transformation $T_{\vec{\lambda},\pi} : J \to J$ which self-induces on the subinterval $J' = \frac{1}{\theta}J$ where $\theta > 1$ is the largest eigenvalue of the Rauzy induction matrix A which is *primitive* ($A^k > 0$, for some $k \in \mathbf{N}$). Let $\vec{\lambda}$ be the largest eigenvector of A and let $\vec{h}$ be the largest eigenvector of A^t. We build a surface above $T_{\vec{\lambda},\pi}$ in the following manner.

For each permutation $\pi \in S_m$, define another permutation σ_π on $\{0, 1, 2, ..., m\}$ by

$$\sigma_\pi(i) = \begin{cases} \pi^{-1}1 - 1, & i = 0, \\ m, & i = \pi^{-1}m, \\ \pi^{-1}(\pi i + 1) - 1, & \text{otherwise.} \end{cases}$$

It can be checked that σ_π is 1:1 on $\{0, 1, 2, ..., m\}$ [Vee82]. Let $\Sigma(\sigma_\pi)$ be the set of cycles of σ_π. Then, there exists an $(|\Sigma(\sigma_\pi)| - 1)$-dimensional subspace of $\mathbf{R}^m_+$ which is the solution space of the linear system of equations

$$h_i - a_i = h_{\sigma_\pi(i)+1} - a_{\sigma_\pi(i)} \tag{2.4}$$

for $0 \le i \le m$ where we put $a_0, h_0, h_{m+1} = 0$.

We build a surface M based on $T_{\vec{\lambda},\pi}$ by gluing m rectangles of dimensions $\lambda_i \times h_i$ together side-by-side with level bases, and "identifying" points on the boundary of the rectangles. The notion of identification means we are associating several points to one single point. The tops of the m rectangles are identified with the base by $s \to T_{\vec{\lambda},\pi}(s)$. A solution vector $\vec{a}$ of (2.4) determines the identifications of the vertical edges by "unzipping" the edges between adjacent rectangles, which is shown in Section 4.1. For four intervals, the interesting case corresponds to the permutation $\pi = (14)(23)$ since it is the only permutation in S_4 that gives rise to a surface whose genus is not 1.

2.4 An Explicit Example of an Interval Exchange Transformation

In this section, we define a particular interval exchange transformation T on an interval J. We show that the transformation T "induces" itself on a subinterval J' of J. This self-induction means T is conjugate to its first return map on the subinterval. In this example, the ratio $\frac{|J|}{|J'|}$ is a non-Pisot number.

Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 3 & 2 & 2 \end{pmatrix}. \tag{2.5}$$

The matrix A is the product of 8 matrices defined by Rauzy induction for 8 steps referenced in Theorem 2.2.1. It begins with the permutation $\pi = (14)(23)$ and follows the induction steps according to the path *bbabaaba*. Since $A^2 > 0$, the Perron-Frobenius Theorem implies there exists a unique eigenvalue $\theta > 0$, larger than the modulus of its conjugates, with corresponding eigenvector $\vec{\lambda} > 0$. In this example $\theta = \frac{7+\sqrt{5}+\sqrt{2}\sqrt{19+7\sqrt{5}}}{4} \approx 4.39026$ and $\vec{\lambda} \approx (0.47726, 0.19967, 0.41836, 1)$. In our example, θ is a non-Pisot number as can be seen in

Figure 2.2 which shows the four roots of the characteristic polynomial $x^4-7x^3+13x^2-7x+1$ of A.

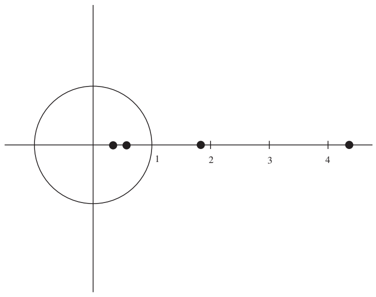

Figure 2.2: Eigenvalues of A.

Using vector $\vec{\lambda}$ and the permutation $\pi = (14)(23)$, we define the interval exchange transformation $T_{\vec{\lambda},\pi} : J \to J$, where $J = [0, \sum_{i=1}^{4} \lambda_i)$. In particular, the interval J is divided into four subintervals J_1, J_2, J_3, J_4 of lengths $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, respectively, which are closed on the left and open on the right. The transformation T maps J onto itself by cutting J into these four subintervals, rearranging their order to J_4, J_3, J_2, J_1, and reattaching the interior endpoints. The explicit formula for $T : J \to J$ is given by

$$Ts = \begin{cases} s + \lambda_2 + \lambda_3 + \lambda_4, & \text{for } s \in J_1 = [0, \lambda_1), \\ s - \lambda_1 + \lambda_3 + \lambda_4, & \text{for } s \in J_2 = [\lambda_1, \lambda_1 + \lambda_2), \\ s - \lambda_1 - \lambda_2 + \lambda_4, & \text{for } s \in J_3 = [\lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \lambda_3), \\ s - \lambda_1 - \lambda_2 - \lambda_3, & \text{for } s \in J_4 = [\lambda_1 + \lambda_2 + \lambda_3, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4). \end{cases}$$

The graph of the transformation T is shown in Figure 2.3.

By inducing eight times, we have an induced transformation on the subinterval $J' = [0, r)$ where $r = \frac{1}{\theta} \sum_{i=1}^{4} \lambda_i$. Note that $\sum_{i=1}^{4} \lambda_i = \theta\lambda_1$, which can be seen by considering the first coordinate of $A\vec{\lambda} = \theta\vec{\lambda}$ are equal. Thus, we can write $J' = [0, \lambda_1) = J_1$. Recall, the induced transformation $T' : J' \to J'$ is defined by $T's = T^{n(s)}s$.

Lemma 2.4.1 *(J', T') is well defined and conjugate to (J, T).*

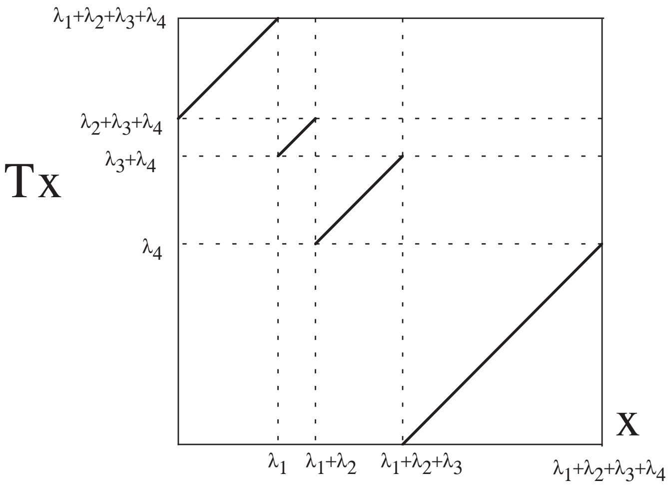

Figure 2.3: Graph of $T : J \to J$.

Proof To show that T' is well defined on all of J', we need $n(s) < \infty$. We partition J' into four subintervals, which we show have constant values of $n(s)$:

$$\begin{aligned} J'_1 &= [0, 4\lambda_1 + \lambda_2 - 2\lambda_4) \\ J'_2 &= [4\lambda_1 + \lambda_2 - 2\lambda_4, 2\lambda_1 + \lambda_2 - \lambda_4) \\ J'_3 &= [2\lambda_1 + \lambda_2 - \lambda_4, 3\lambda_1 + 2\lambda_2 + \lambda_3 - 2\lambda_4) \\ J'_4 &= [3\lambda_1 + 2\lambda_2 + \lambda_3 - 2\lambda_4, \lambda_1). \end{aligned}$$

Note that the subintervals J'_i have lengths λ'_i where $\vec{\lambda}' = A^{-1}\vec{\lambda} = \frac{1}{\theta}\vec{\lambda}$ comes from the Rauzy induction.

To see that each subinterval returns to J', we build a tower of intervals, called the *Kakutani skyscraper* (Figure 2.4), by stacking the images of each subinterval above itself until one of its iterates returns to J'. For example, it is easy to see

$T(J'_1) = T([0, 4\lambda_1 + \lambda_2 - 2\lambda_4)) = [\lambda_2 + \lambda_3 + \lambda_4, 4\lambda_1 + 2\lambda_2 + \lambda_3 - \lambda_4) \subset J_4$,

$T^2(J'_1) = T([\lambda_2 + \lambda_3 + \lambda_4, 4\lambda_1 + 2\lambda_2 + \lambda_3 - \lambda_4)) = [-\lambda_1 + \lambda_4, 3\lambda_1 + \lambda_2 - \lambda_4) \subset J_2$,

$T^3(J'_1) = [-2\lambda_1 + \lambda_3 + 2\lambda_4, 2\lambda_1 + \lambda_2 + \lambda_3) \subset J_4$, and

$T^4(J'_1) = [-3\lambda_1 - \lambda_2 + 2\lambda_4, \lambda_1) \subset J'$.

Thus, the fourth iterate of J'_1 returns to J', and it follows that $n(s) = 4$ for all $s \in J'_1$.

Note that in the skyscraper, the tower above J_1' has 4 levels, which we label J_1, J_4, J_2, J_4 according to the subintervals of J which contain the iterates of J_1'. By a similar calculation,

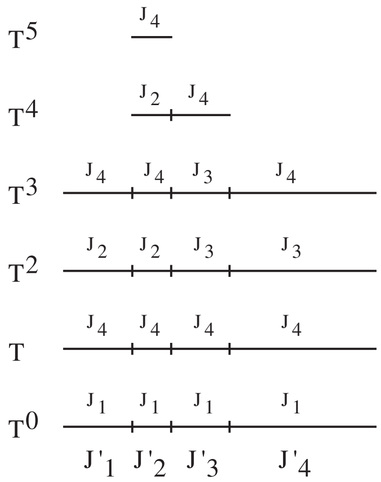

Figure 2.4: Kakutani skyscraper of J'.

one can show $n(s) = 6$ for all $s \in J_2'$, $n(s) = 5$ for all $s \in J_3'$, and $n(s) = 4$ for all $s \in J_4'$. In Figure 2.4, we show the corresponding labeling of all the levels above the base subintervals. It follows that T' is well defined for all $s \in J'$. In fact, T' is an interval exchange. As we see below, it exchanges the intervals J_1', J_2', J_3', and J_4' according to the same permutation π.

To show that (J', T') is conjugate to (J, T), we define the conjugacy $g : J \to J'$ by $g(s) = \frac{1}{\theta}s$ for each $s \in J$. Obviously, g is one-to-one, onto, and continuous. We must also show that g satisfies the commutation relation $g \circ T = T' \circ g$. For example given $s \in J_1$ we have

$$g(Ts) = g(s + \lambda_2 + \lambda_3 + \lambda_4) = \tfrac{1}{\theta}(s + \lambda_2 + \lambda_3 + \lambda_4) = \tfrac{1}{\theta}s + \tfrac{1}{\theta}(\lambda_2 + \lambda_3 + \lambda_4)$$

and

$$T'(g(s)) = T'(\tfrac{1}{\theta}s) = T^4(\tfrac{1}{\theta}s) = \tfrac{1}{\theta}s - 3\lambda_1 - \lambda_2 + 2\lambda_4.$$

But $\frac{1}{\theta}(\lambda_2+\lambda_3+\lambda_4) = -3\lambda_1-\lambda_2+2\lambda_4$, as can be seen by summing the last three coordinates of $A^{-1}\vec{\lambda} = \frac{1}{\theta}\vec{\lambda}$ where

$$A^{-1} = \begin{pmatrix} 4 & 1 & 0 & -2 \\ -2 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ -2 & -2 & -1 & 2 \end{pmatrix}.$$

A similar argument is used for $s \in J_2, J_3, J_4$. The fact that the induced interval exchange transformation T' is also defined by the permutation π follows from the conjugacy $g \circ T = T' \circ g$. $\Box$

An interval exchange transformation T on J is *minimal*, if the orbit $\mathcal{O}(s) = \{T^i s : i \in \mathbf{N}\}$ of each point $s \in J$ is dense in J [Vee82]. Using the properties of π and $\vec{\lambda}$, we show that T is minimal.

Lemma 2.4.2 *The coordinates of $\vec{\lambda}$ are rationally independent.*

Proof Let $\{\omega_1, \omega_2, \omega_3, \omega_4\} = \{1, \sqrt{5}, \sqrt{2}\sqrt{19+7\sqrt{5}}, \sqrt{10}\sqrt{19+7\sqrt{5}}\}$. By an explicit calculation, we can write $\lambda_i = c_{i1}\omega_1 + c_{i2}\omega_2 + c_{i3}\omega_3 + c_{i4}\omega_4$ for each $i = 1,2,3,4$ where $c_{ij} \in \mathbf{Q}$. The numbers $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ are easily seen to be rationally independent. We can write $\vec{\lambda} = \vec{\omega}C$, where $\vec{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4)$ and $C = (c_{ij})$ is given by

$$C = \begin{pmatrix} -\frac{1}{4} & \frac{5}{4} & -\frac{7}{4} & 1 \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & 0 \\ -\frac{1}{8} & \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{8} & -\frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}.$$

If $\vec{\lambda} \cdot \vec{n} = 0$, then $(c_{i1}, c_{i2}, c_{i3}, c_{i4}) \cdot \vec{n} = 0$ for each $i = 1,2,3,4$ since the coordinates of $\vec{\omega}$ are rationally independent. Written in matrix form, $C\,\vec{n}^t = 0$, which implies $\vec{n} = 0$ since C is invertible. $\Box$

Lemma 2.4.3 *T is minimal on J.*

Proof Suppose $\vec{\lambda}$ has been scaled to represent the unit interval ($\sum_{i=1}^{4} \lambda_i = 1$). Then, the only rational relations among the scaled $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are multiples of $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$ which follows from Lemma 2.4.2. Combined with π being irreducible, this property makes T minimal, as shown in M. Keane [Kea75]. $\square$

Chapter 3

Substitution Systems

3.1 Definitions

In this chapter we construct a *substitution system* on an alphabet $\mathcal{A} = \{1, 2, 3, 4\}$ and use it to show that T is *uniquely ergodic*, meaning there is only one T-invariant measure. Let $\mathcal{A}^*$ be the set of finite words in the alphabet $\mathcal{A}$, and let $\mathcal{A}^{\mathbf{N}}$ be the set of (one-sided) sequences with values in $\mathcal{A}$, endowed with the product topology. We define a substitution map $S : \mathcal{A} \to \mathcal{A}^*$ by

$$1 \to 1424 \qquad 2 \to 142424 \qquad 3 \to 14334 \qquad 4 \to 1434. \tag{3.1}$$

This substitution records the subintervals of J which lie over each J_i' in the Kakutani skyscraper (Figure 2.4). In particular, if $i \to i_1 i_2 ... i_k$, then the levels over J_i are labeled $J_{i_1}, J_{i_2}, ..., J_{i_k}$ in the tower.

By a standard construction [Que87], we can extend S to be a map $S : \mathcal{A}^{\mathbf{N}} \to \mathcal{A}^{\mathbf{N}}$ defined by $S(\varpi) = S(\varpi_1)S(\varpi_2)...$ where $\varpi = \varpi_1\varpi_2...$ and each $\varpi_i \in \mathcal{A}$. Note that $\mathcal{A}^{\mathbf{N}}$ is compact and metrizable in the product topology. A standard metric for the product topology [LM95] is

$$d(x, y) = \sum_{\{i \in \mathbf{N} : x_i \neq y_i\}} \frac{1}{2^i}.$$

Lemma 3.1.1 *S is a contraction with respect to the metric d.*

The map S is a contraction since the first two digits of all the substitutions are the same, namely 12. (Also, note that because each substitution has length at least 4 digits, the first place two strings differ is the $(4n+2)$-position after applying the substitution map, where n is the first place the two strings differ before applying the mapping S.)

By the Contraction Mapping Theorem, S has a unique fixed point $u \in \mathcal{A}^{\mathbf{N}}$. This fixed point u is the limit of the sequence of words $\{S^n(1)\}$. Define X to be the orbit closure of $\mathcal{O}(u)$ in $\mathcal{A}^{\mathbf{N}}$, namely $cl(\mathcal{O}(u))$, where $\mathcal{O}(u) = \{\sigma^i u : i \in \mathbf{N}\}$ is the orbit of u under the shift map $\sigma : \mathcal{A}^{\mathbf{N}} \to \mathcal{A}^{\mathbf{N}}$ defined by $(\sigma x)_i = x_{i+1}$.

Lemma 3.1.2 *(X, σ) is uniquely ergodic.*

Proof Let $B = (b_{ij})$ be the 4×4 *substitution matrix* where $b_{i,j}$ is the number of times i appears in the substitution for j. Then we have that $B = A$. Since $A^2 > 0$, (X, σ) is uniquely ergodic by a standard result from the theory of substitutions [Que87]. □

3.2 Conjugacy

Lemma 3.2.1 *There exists an injective, right continuous mapping $\phi : J \to \mathcal{A}^{\mathbf{N}}$ such that $\phi(J)$ is σ-invariant and $\phi(J)$ is dense in X.*

Proof We define a mapping $\phi : J \to \mathcal{A}^{\mathbf{N}}$ by

$$(\phi(s))_i = k, \quad \text{where} \quad T^i s \in J_k, \quad \text{for each} \quad i \in \mathbf{N}. \tag{3.2}$$

Since each subinterval J_k is open on the right, ϕ is right continuous.

Now, suppose there exist two points $s_1, s_2 \in J$ with $s_1 < s_2$ and $\phi(s_1) = \phi(s_2)$. Then, $s_1, s_2 \in J_n$ for some $n = 1, 2, 3, 4$ since $(\phi s_1)_0 = (\phi s_2)_0$.

Let c_1, c_2, c_3, c_4 be the left endpoints of the subintervals of J. By [Kea75], we know that T^{-1} is also minimal. Thus, there exists $n_i \geq 0$ with $T^{-n_i}c_i$ in the open interval (s_1, s_2) for each $i = 1, 2, 3, 4$. Let c_k be the first endpoint with $T^{-n_k}c_k \in (s_1, s_2)$.

Since T is 1:1, $T^{n_k}s_1 \neq T^{n_k}s_2$. Suppose $T^{n_k}s_1 < T^{n_k}s_2$, which occurs when $k = 2, 3,$ or 4. Then, $(\phi s_1)_{n_k} = 1, 2, ...,$ or $k-1$ and $(\phi s_2)_{n_k} = k, k+1, ...,$ or 4. Suppose $T^{n_k}s_1 > T^{n_k}s_2$, which occurs when $k = 1, 2,$ or 3. Then, $(\phi s_1)_{n_k} = 4$ and $(\phi s_2)_{n_k} = 1, 2,$ or 3. This shows $(\phi s_1)_{n_k} \neq (\phi s_2)_{n_k}$ which is a contradiction.

To show $\phi \circ T = \sigma \circ \phi$, we show $(\phi(Ts))_i = (\sigma(\phi s))_i$ for all $s \in J$ and $i \in \mathbf{N}$. Suppose $(\phi(Ts))_i = k$. Then, $T^i(Ts) = T^{i+1}s \in J_k$, so $k = (\phi s)_{i+1} = (\sigma(\phi s))_i$, and $\phi(J)$ is σ-invariant.

Finally, we need only to show that $\phi(J) \subset X$. First we show $\phi(0) = u$, the unique fixed point of the substitution S. Define mapping $\phi' : J' \to \mathcal{A}^{\mathbf{N}}$ by

$$(\phi'(s))_i = k, \quad \text{where} \quad (T')^i s \in J'_k, \quad \text{for each} \quad i \in \mathbf{N}.$$

Since (J, T) is conjugate to (J', T') by $g : J \to J'$, the dynamics of T on points in J is the same as the dynamics of T' on points in J'. Thus, we have $\phi(s) = \phi'(g(s))$ for all $s \in J$. As mentioned earlier, the substitution S records the subintervals of J the orbit of $s \in J'$ visits, until it returns to J'. We claim $S(\phi'(s)) = \phi(s)$ for all $s \in J'$, which we show by induction on the n^{th} digit of $\phi'(s)$. We separate the points of J' into the following four cases.

$s \in J'_1 \Rightarrow S((\phi' s)_0) = S(1) = 1424 = (\phi s)_{[0,n(s)-1]}$

$s \in J'_2 \Rightarrow S((\phi' s)_0) = S(2) = 142424 = (\phi s)_{[0,n(s)-1]}$

$s \in J'_3 \Rightarrow S((\phi' s)_0) = S(3) = 14334 = (\phi s)_{[0,n(s)-1]}$

$s \in J'_4 \Rightarrow S((\phi' s)_0) = S(4) = 1434 = (\phi s)_{[0,n(s)-1]}$

Assuming $S((\phi' s)_{[0,m-1]}) = (\phi s)_{[0,n^m(s)-1]}$, we show $S((\phi' s)_{[0,m]}) = (\phi s)_{[0,n^{m+1}(s)-1]}$. Note that $T's = T^{n(s)}s \in J'$ for all $s \in J'$, so we can write $(T')^m s = T^{n^m(s)}s$ for all $s \in J'$ and $m \in \mathbf{N}$. We apply $(T')^m s$ in each above case to obtain $S((\phi'(T')^m s)_0) = (\phi T^{n^m(s)} s)_{[0,n(s)-1]}$. But $S((\phi'(T')^m s)_0) = S((\phi' s)_m)$ and $(\phi T^{n^m(s)} s)_{[0,n(s)-1]} = (\phi s)_{[n^m(s),n^{m+1}(s)-1]}$. Using this fact and the assumption, we have $S((\phi' s)_{[0,m]}) = (\phi s)_{[0,n^{m+1}(s)-1]}$ which proves the claim.

Finally, we note that $S(\phi(0)) = S(\phi'(g(0))) = S(\phi'(0)) = \phi(0)$. Consequently, $\phi(0)$ is the fixed point of S, so $\phi(0) = u$. Thus, 0 and all of its iterates $\{T^i 0\}$ have their ϕ-images in X. By the minimality of T, for every $s \in J$, there exists a sequence $\{n_j\} > 0$ with $\{T^{n_j}0\} \searrow s$. Thus, by right continuity of ϕ, we have $\{\phi(T^{n_j}0)\} = \{\sigma^{n_j}(\phi(0)\} = \{\sigma^{n_k}u\} \to \phi(s)$, so $\phi(s) \in cl(\mathcal{O}(u)) = X$. $\square$

Note that ϕ is also left continuous on J except at the pre-images of endpoints of the subintervals of J. Let $J^o = J \setminus \cup_{i=0}^{\infty} T^{-i}(\{0, \lambda_1, \lambda_1+\lambda_2, \lambda_1+\lambda_2+\lambda_3\})$ be the subset of J which excludes all pre-images of endpoints of the subintervals.

Theorem 3.2.1 *There exists a continuous mapping $\rho : (X, \sigma) \to (J, T)$ which is onto and 1:1 on the complement of a countable dense set and 2:1 elsewhere. Also, $\phi^{-1} = \rho$ and ρ satisfies the commutation condition $\rho\sigma = T\rho$ for $x \notin \rho^{-1}(\{\lambda_1, \lambda_1+\lambda_2, \lambda_1+\lambda_2+\lambda_3\})$. In particular, $card\{\rho^{-1}(s)\} = 2$, for $s \in J \setminus J^o$.*

Proof Define the map $\rho : X \to J$ by $\rho(\sigma^k u) = T^k 0$ for each $k \in \mathbf{N}$. We claim ρ is uniformly continuous on $\mathcal{O}(u)$. Consider $x = \sigma^k u$ for some $k \in \mathbf{N}$. Then, for each $l \in \mathbf{N}$ we have $x = .u_k u_{k+1} ... u_{k+l+1} ...$, which means $\rho(\sigma^k u) = T^k(0) \in J_{u_k}$. So, $T \circ \rho(\sigma^k u) = T^{k+1}(0) \in J_{u_{k+1}}$, which gives $\rho(\sigma^k u) \in T^{-1}(J_{u_{k+1}})$. It follows that $\rho(\sigma^k u) \in T^{-l}(J_{u_{k+l}})$. Thus, we can write

$$\rho(\sigma^k u) \in \bigcap_{j=0}^{l} T^{-j}(J_{u_{k+j}}).$$

For each $l \in \mathbf{N}$, define

$$\epsilon_l = \max_{[u_k ... u_{k+l}]} \{\mathrm{diam}[\bigcap_{j=0}^{l} T^{-j}(J_{u_{k+j}})]\}$$

over all words $[u_k ... u_{k+l}]$ of length l. By minimality of T, $\epsilon_l \to 0$ as $l \to \infty$.

Let $\epsilon > 0$. Choose $L > 0$ such that $\epsilon_l < \epsilon$ for $l \geq L$. Then, for any $\sigma^{k_1} u, \sigma^{k_2} u \in \mathcal{O}(u)$ with $u_{k_1} ... u_{k_1+l} = u_{k_2} ... u_{k_2+l}$ we have $\rho(\sigma^{k_1} u), \rho(\sigma^{k_2} u) \in \cap_{j=0}^{l} T^{-j}(J_{u_{k_1+j}})$. But this set has diameter less than $\epsilon_l < \epsilon$, so $|\rho(\sigma^{k_1} u) - \rho(\sigma^{k_2} u)| < \epsilon$. Therefore, we can extend ρ to a continuous map on X since $\mathcal{O}(u)$ is dense in X.

To prove $\rho = \phi^{-1}$, it suffices to show $\rho(\phi(s)) = s$ on J. Let $s \in J$. Since T is minimal, there exists an increasing sequence $\{n_i\}$ such that $\{T^{n_i}(0)\} \searrow s$. Since ϕ is right continuous, $\{\phi(T^{n_i}(0))\} \to \phi(s)$. But $\phi(T^{n_i}(0)) = \sigma^{n_i} u$ for each n_i, so $\{\sigma^{n_i} u\} \to \phi(s)$. Therefore, by continuity of ρ,

$$\rho(\phi(s)) = \rho(\lim_{n_i \to \infty} \sigma^{n_i} u) = \lim_{n_i \to \infty} \rho(\sigma^{n_i} u) = \lim_{n_i \to \infty} T^{n_i}(0) = s$$

by definition of s. $\square$

To make X the union of images from J, we define another interval exchange transformation $\widetilde{T} : \widetilde{J} \to \widetilde{J}$ where $\widetilde{J} = (0, \sum_{i=1}^{4} \lambda_i]$. The mapping $\widetilde{T}$ permutes the left open subintervals $\widetilde{J}_1, \widetilde{J}_2, \widetilde{J}_3, \widetilde{J}_4$ exactly as T. The associated mapping $\widetilde{\phi} : \widetilde{J} \to \mathcal{A}^{\mathbf{N}}$ is injective and left continuous. We have $\phi(J) \cup \widetilde{\phi}(\widetilde{J}) = X$, $\phi(J) \cap \widetilde{\phi}(\widetilde{J}) = \phi(J^o)$. The set of singular points is $\phi(J) \triangle \widetilde{\phi}(\widetilde{J})$.

Theorem 3.2.2 *The transformation (J, T) is uniquely ergodic.*

This follows from Lemma 3.1.2, Lemma 3.2.1, and Theorem 3.2.1 since ϕ is a measurable bijection except on a countable dense set.

In Chapter 5, we consider the future and past images of $s \in J$. Thus, it is important to note that we can extend one-sided sequences in $X \subset \mathcal{A}^{\mathbf{N}}$ to two-sided sequences in $\mathcal{A}^{\mathbf{Z}}$. This extension is unique for ϕ-images of points in J whose inverse images do not included any endpoints of subintervals. For points in $s \in J$ with $T^{-n}s \in \{0, \lambda_1, \lambda_1+\lambda_2, \lambda_1+\lambda_2+\lambda_3\}$, there exist two left extensions in the shift space.

Chapter 4

Diffeomorphisms

4.1 Definitions

In this chapter we construct a diffeomorphism D on a surface M. The surface is defined as the phase space of the suspension of the interval exchange transformation T on J, under a height function $\widehat{h} : J \to \mathbf{R}$, which depends on the matrix A. Note that the Perron-Frobenius eigenvalues of A and A^t are the same, namely θ. The function $\widehat{h}$ is defined in terms of the Perron-Frobenius eigenvector $\vec{h}$ of A^t, namely $\widehat{h}(s) = h_i$ for $s \in J_i$, $i = 1, 2, 3, 4$. Here $\vec{h} \approx (1.09529, 1.71333, 1.29496, 1)$.

Define the surface M by

$$M = \{(s, y) \in \mathbf{R}^2 : s \in J, y \in [0, \widehat{h}(s))\}$$

with the following identifications. We eliminate the the horizontal edges by identifying the top of M to its base by transformation T. In particular, we equate the points $(s, \widehat{h}(s))$ and $(Ts, 0)$. This is as shown by the different horizontal lines in Figure 4.1.

We introduce "zippers" to define the identifications of the vertical edges of the polygon to obtain a closed surface. The zippers begin at each of the top corners, except the left, of the polygon and unzip the vertical edges down to a specified height given by $\vec{a} \in \mathbf{R}^4$. The

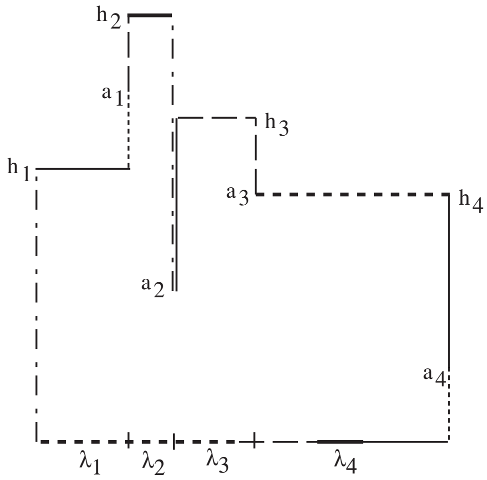

Figure 4.1: Surface M

vertical edges of the rectangles are also identified by different kinds of lines in Figure 4.1. Recall vector $\vec{a}$ is in a $(|\Sigma(\sigma_\pi)|-1)$-dimensional subspace that satisfies system (2.4). As shown later, the identifications by $\vec{a}$ are invariant with respect to D. To satisfy this invariance, it follows from [Vee82] that $\vec{a}$ should be the solution to the matrix equation $L\vec{h}+\vec{a}=\theta\vec{a}$, where the matrix L, described after Lemma 4.1.1, is given by

$$L=\begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In this example $\vec{a}\approx(1.41836, 0.61803, 1, 0.32307)$.

We define the hyperbolic map D on M by constructing a second realization of M. First, we create a partition $\mathcal{Q}$ of M by dividing M into 11 rectangles as shown in the polygon on the left side of Figure 4.2. The partition $\mathcal{Q}=\{Q_1,Q_2,...,Q_{11}\}$ is defined by

$Q_i = M \cap ([\alpha_{i-1}, \alpha_i) \times \mathbf{R})$, for $i = 1, 2, ..., 11$, where the α_i are given by

$$\begin{array}{lcllcl}
\alpha_0 & = & 0 & \alpha_6 & = & \lambda_1 + \lambda_2 + \lambda_3 \\
\alpha_1 & = & \lambda_1 & \alpha_7 & = & -\lambda_1 - \lambda_2 + 2\lambda_4 \\
\alpha_2 & = & -\lambda_1 + \lambda_4 & \alpha_8 & = & \lambda_3 + \lambda_4 \\
\alpha_3 & = & \lambda_1 + \lambda_2 & \alpha_9 & = & -2\lambda_1 + \lambda_3 + 2\lambda_4 \\
\alpha_4 & = & 2\lambda_1 + 2\lambda_2 + \lambda_3 - \lambda_4 & \alpha_{10} & = & \lambda_2 + \lambda_3 + \lambda_4 \\
\alpha_5 & = & \lambda_4 & \alpha_{11} & = & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4.
\end{array}$$

Using this partition, we restack the 11 rectangles using $J' = J_1$ as the new base interval. This new surface is denoted M' and is shown in the center of Figure 4.2.

Lemma 4.1.1 *The identifications of M are preserved in M'.*

Proof The horizontal identifications are preserved because T induces itself on $J' = J_1$. Thus, we can stack the rectangles of the partition $\mathcal{Q}$ above Q_1 to construct M' so that the tops of lower rectangles agree with the bases of stacked rectangles as shown.

To see that the vertical identifications are preserved, we label the vertical sides of each member of $\mathcal{Q}$ by the letters $\{\mathtt{a}, \mathtt{b}, ..., \mathtt{m}\}$ as shown on the first construction of M. It suffices to show that each lettered side adjoins the same rectangles of $\mathcal{Q}$ on the two polygons.

Consider the vertical edge labeled `a` in the first polygon, namely M. Edge `a` is the right edge of Q_3 above edge `f` and the left edge of Q_1. This is also true for the middle polygon, representing M'. Considering edge `b`, it adjoins Q_1 on its left to Q_2 on its right in both diagrams. This argument can be continued for each vertical edge.

Therefore, each rectangle in $\mathcal{Q}$ is identified in the same way to the other rectangles in both polygons. Thus, the surface M' is a second realization of M. □

Note that the diagram of M' in Figure 4.2 can also be viewed as the diagram of M with different coordinates. Moreover, these surfaces are actually homeomorphic. The second construction has its base shrunk by a factor of $\frac{1}{\theta}$ and its height stretched by a

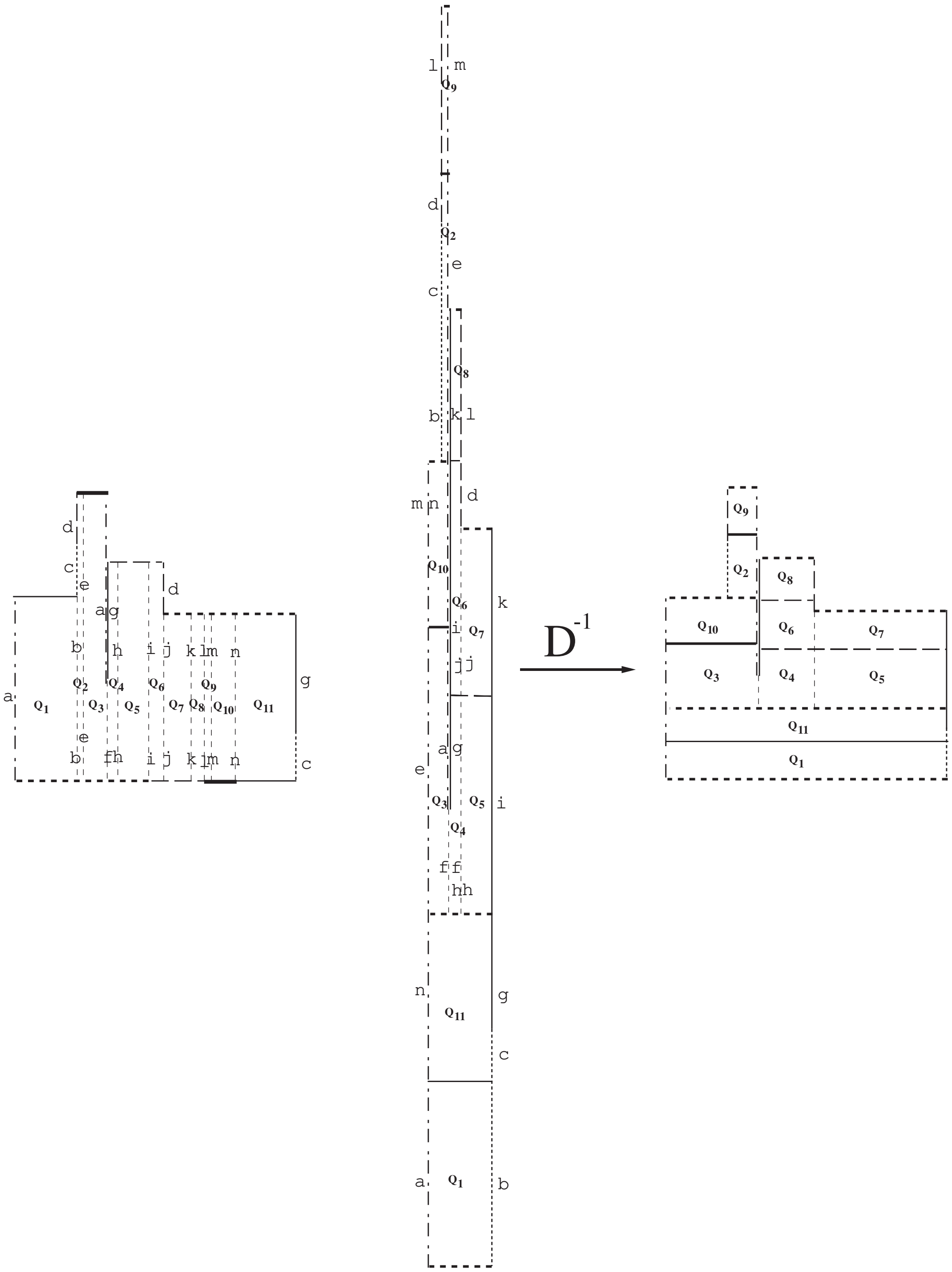

Figure 4.2: Map $D : M \to M$ after scaling.

factor of θ. This follows since $|J_1| = \frac{1}{\theta}|J|$, $\theta h_1 = h_1 + h_2 + 2h_4$, $\theta h_2 = h_1 + 2h_2 + 3h_4$, $\theta h_3 = h_1 + 2h_3 + 2h_4$, and $\theta h_4 = h_1 + h_3 + 2h_4$ which can be seen by considering $A^t\vec{h} = \theta\vec{h}$.

The matrix L is defined by the stacking of rectangles. Each row of L is determined by the heights of the rectangles stacked below $\vec{a}$ in the new construction of M. We use these properties to define a diffeomorphism $D : M \to M$ in which D^{-1} sends (s, y) in M' to $(\theta s, \frac{1}{\theta}y)$ in M.

Lemma 4.1.2 *The map D is a diffeomorphism on M.*

4.2 Pseudo-Anosov Diffeomorphism

To define the notion of a pseudo-Anosov diffeomorphism, we must first introduce the idea of measured foliations. For any surface M a *measurable foliation* $\mathcal{F}$ on M is a foliation, possibly with a finite number of singularities, endowed with an invariant transverse measure [Thu88]. For each nonsingular point on M there is a neighborhood U and chart $\varphi : U \to \mathbf{R}^2$ such that $\varphi^{-1}(\mathbf{R} \times y_*)$ are the leaves of $\mathcal{F}$, for each $y_* \in \mathbf{R}$. So, the chart φ sends the leaves of $\mathcal{F}$ to horizontal lines in $\mathbf{R}^2$. If U_i and U_j intersect, then there exists a transition function φ_{ij} such that $\varphi_{ij} \circ \varphi_j = \varphi_i$ on $U_i \cap U_j$ and φ_{ij} is of the form

$$\varphi_{ij}(x, y) = (f(x, y), c_{ij} \pm y)$$

where c_{ij} is a constant. The allowed singularities are p-pronged saddle points where $p \geq 3$.

Consider the foliation $\mathcal{F}$ of M consisting of vertical lines along M, where M is viewed as the polygon in Figure 4.1. All regular points of M have neighborhoods which look like Figure 4.3. There is one singularity of the foliation $\mathcal{F}$ whose neighborhood looks like Figure 4.4.

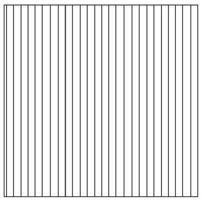

Figure 4.3: Neighborhood of regular points of foliation $\mathcal{F}$.

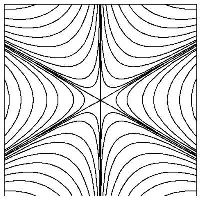

Figure 4.4: Neighborhood of the singularity of foliation $\mathcal{F}$.

To determine the genus of surface M, we analyze the singularity of the foliation $\mathcal{F}$. If we begin at the number 1, located in the middle of the diagram in Figure 4.5 and rotate counter-clockwise through the increasing numbers until we return to the number 1 located in the lower left corner, then we travel through an angle of 6π.

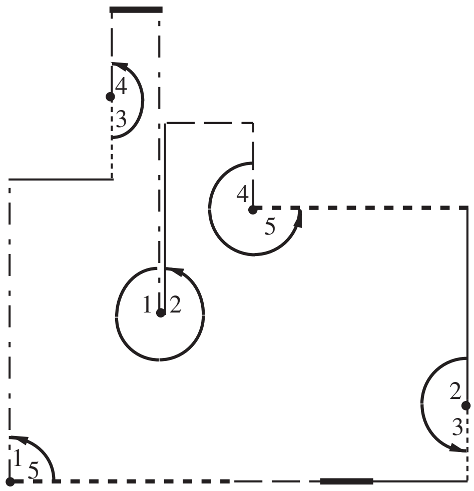

Figure 4.5: Genus of surface M.

Thus, the point we have encircled is a singularity of the foliation, in particular a 6-pronged saddle point. Equivalently, this singularity is also called a *branch point* of the surface M of order 3. All other points on M have a total angle of 2π, so this point is the only branch point of M. We can compute the genus of M by using the following equation

[Vee82].

$$2g(M) - 2 \quad = \sum_{branch\ points} (\text{order of branch point} - 1)$$

Thus, we have $g(M) = 2$ in our case.

A *pseudo-Anosov diffeomorphism* D is a homeomorphism of a surface such that there exists a number $\theta > 1$ and a pair of transverse measurable foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ such that $D(\mathcal{F}^s) = \frac{1}{\theta}\mathcal{F}^s$ and $D(\mathcal{F}^u) = \theta\mathcal{F}^u$ [Thu88]. This means, in particular, that D preserves the two foliations $\mathcal{F}^s$ and $\mathcal{F}^u$, and contracts the distance between points on a leaf of $\mathcal{F}^s$ by a factor of $\frac{1}{\theta}$ and expands the distance between points on a leaf of $\mathcal{F}^u$ by a factor of θ.

Theorem 4.2.1 *The mapping $D : M \to M$ is a pseudo-Anosov diffeomorphism.*

Proof It is obvious from Figure 4.2 that D expands the (vertical) leaves of $\mathcal{F} = \mathcal{F}^u$ by a factor of θ. This occurs because D stretches the y-coordinates of points on the surface by a factor of θ. Next, if we consider another foliation $\mathcal{F}'$ whose leaves are defined to be horizontal lines across the surface, then we have $\mathcal{F}' = \mathcal{F}^s$. The map D contracts its leaves by a factor of $\frac{1}{\theta}$. Similarly, this occurs because D contracts the x-coordinates of points on the surface by a factor of $\frac{1}{\theta}$. $\square$

Now we construct a Markov partition for D on M. A *Markov partition* [GH83] for a hyperbolic diffeomorphism $D : M \to M$ is a collection $\mathcal{P} = \{P_1, \ldots, P_k\}$ of rectangles (closed subsets whose four sides are segments of the stable and unstable foliations) such that:

1) $\bigcup_{i=1}^{k} P_i = M$;

2) $int(P_i) \bigcap int(P_j) = \emptyset$ for all $i \neq j$;

3) If $z \in int(P_i)$ and $D(z) \in int(P_j)$, then $D(\mathcal{F}^s(z, P_i)) \subset \mathcal{F}^s(D(z), P_j)$ and $D(\mathcal{F}^u(z, P_i)) \supset \mathcal{F}^u(D(z), P_j)$.

We note that Markov partitions exist for all pseudo-Anosov diffeomorphisms [FS79]. In our example, $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$ is the partition of M shown in Figure 4.6.

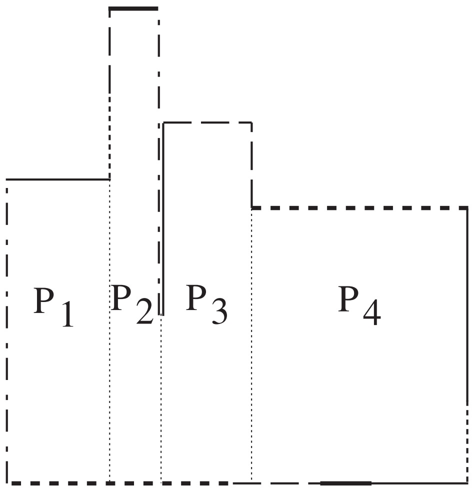

Figure 4.6: Markov partition $\mathcal{P}$

Lemma 4.2.1 *$\mathcal{P}$ is a Markov partition of M for D.*

Proof In our example (1) and (2) are obvious. Condition (3) is true because $\mathcal{F}^s$ and $\mathcal{F}^u$ are strictly horizontal and vertical, respectively, and D stretches/contracts the surface M vertically/horizontally. Because the vertical edges of each member of $\mathcal{P}$ begin at the base of M and extend to the top of M, as shown in Figure 4.7, we are assured that $D(\mathcal{F}^u(z, P_i)) \supset \mathcal{F}^u(D(z), P_j)$. Also, the image of each $\mathcal{F}^s(z, P_i)$ is contained in a member of the partition $\mathcal{Q}$ which is strictly contained in a member of $\mathcal{P}$, so we have $D(\mathcal{F}^s(z, P_i)) \subset \mathcal{F}^s(D(z), P_j)$. This can be seen in Figure 4.8. $\square$

4.3 Suspension Flow

Now we define a flow F^t on the surface M to be the flow that moves along the leaves of the vertical foliation $\mathcal{F}$ with unit speed. The trajectory of a point along the flow F^t will define

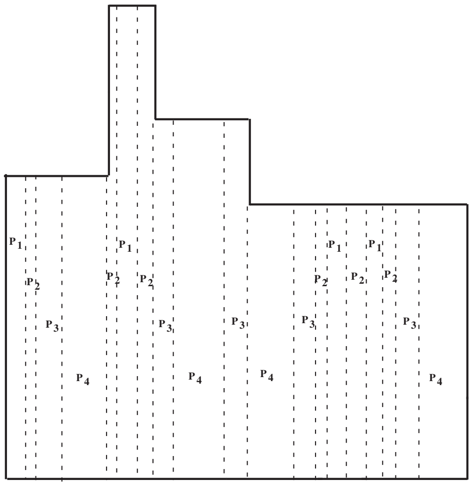

Figure 4.7: Image of partition $\mathcal{P}$.

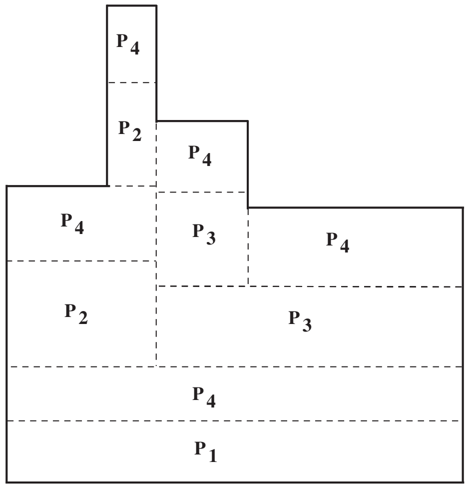

Figure 4.8: Inverse image of partition $\mathcal{P}$.

a tiling in the next chapter. The explicit definition for the flow $F^t : M \to M$ is given by

$$F^t(s,y) = \begin{cases} (T^{\widetilde{m}(s,t+y)+1}s, t+y-\sum_{i=0}^{\widetilde{m}(s,t+y)} \widehat{h}(T^i s)), & \text{for } t \geq \widehat{h}(s)-y, \\ (s, y+t), & \text{for } -y \leq t < \widehat{h}(s)-y, \\ (T^{-\widetilde{m}(s,t+y)}s, t+y+\sum_{i=1}^{\widetilde{m}(s,t+y)} \widehat{h}(T^{-i}s)), & \text{for } t < -y \end{cases} \tag{4.1}$$

where $\widetilde{m}(s,r) : J \times \mathbf{R} \to \mathbf{Z}$ is the integer such that

$$\widetilde{m}(s,r) = \begin{cases} \sum_{i=0}^{\widetilde{m}} \widehat{h}(T^i s) \leq r < \sum_{i=0}^{\widetilde{m}+1} \widehat{h}(T^i s), & \text{for } r \geq \widehat{h}(s), \\ \sum_{i=1}^{\widetilde{m}-1} \widehat{h}(T^{-i} s) < -r \leq \sum_{i=1}^{\widetilde{m}} \widehat{h}(T^{-i}s), & \text{for } r < 0. \end{cases}$$

In particular, the flow is the suspension of T with return time function $\widehat{h}$. Note that this flow F^t fails to be defined on the singular leaf of $\mathcal{F}$.

Chapter 5

Tiling Dynamical Systems

5.1 Definitions

In this chapter, we introduce 1-dimensional tiling dynamical systems. In particular, points of this system are tilings of $\mathbf{R}$, which we regard as partitions of $\mathbf{R}$ into intervals with finitely many different lengths. We consider the space $\widetilde{\mathcal{U}}$ of such tilings associated with our construction. Most of the tilings in this space are determined by the points on the surface M and, in particular, their trajectories by the flow through the Markov partition $\mathcal{P}$.

First, we give some general definitions for tilings of $\mathbf{R}^n$. A *tile* I of $\mathbf{R}^n$ is a subset of $\mathbf{R}^n$ that is the closure of its interior. The *equivalence class* of a tile is the set of all tiles which are translations of the tile. A *tiling* $\mathcal{U}$ of $\mathbf{R}^n$ is a set of tiles which covers $\mathbf{R}^n$ ($\cup\{I : I \in \mathcal{U}\} = \mathbf{R}^n$) such that the tiles belong to only a finite number of equivalence classes, they have mutually disjoint interiors and each compact set in $\mathbf{R}^n$ intersects only a finite number of tiles. For a tiling of $\mathbf{R}$, the tiles are intervals with a finite number of finite lengths. An example of a *patch*, finite set of adjoining tiles, of a 2-dimensional tiling is shown in Figure 5.1. A tiling $\mathcal{U}$ is said to have the *local isomorphism property* if, for any $\mathcal{U}$-patch P, there exists $R_P > 0$ such that every open ball $B_{R_P}(y)$ contains a $\mathcal{U}$-patch equivalent to P. So, if we chose a patch and look through a large enough window anywhere

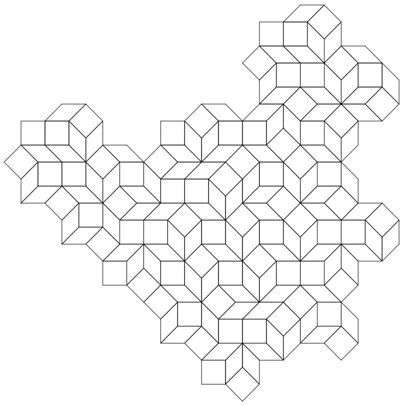

Figure 5.1: Patch of a tiling of $\mathbf{R}^2$.

in the tiling, we see our chosen patch. A tiling has a *finite number of local patterns* if for each $r > 0$ there exist only finitely many patches of diameter less than r. So, through windows of a chosen diameter, we see only a finite number of patches throughout the tiling.

We can define a metric on the set of tilings $\widetilde{\mathcal{U}}$ (with the same set of tiles) in which two tilings are close if a large patch of their tiles agree in a neighborhood of the origin, up to a small translation. This metric [Sol97] is given by

$$m(\mathcal{U}_1,\mathcal{U}_2) = min\{1, \widetilde{m}(\mathcal{U}_1,\mathcal{U}_2)\} \tag{5.1}$$

where

$$\begin{aligned}\widetilde{m}(\mathcal{U}_1,\mathcal{U}_2) = \inf\{\epsilon > 0 : &\text{ there exists a } \mathcal{U}_1\text{-patch } P_1 \text{ and a } \mathcal{U}_2\text{-path } P_2 \text{ such that} \\ &P_1, P_2 \supset B_{\frac{1}{\epsilon}}(0) \text{ and } P_1 = P_2 + \vec{g}, \text{ for some } \vec{g} \in \mathbf{R}^n \text{ where } \|\vec{g}\| \le \epsilon\}.\end{aligned}$$

Given a tiling $\mathcal{V}$, we can define a tiling space $\widetilde{\mathcal{V}} = cl\{\sigma^{\vec{t}}(\mathcal{V}) : \vec{t} \in \mathbf{R}^n\}$ where $\sigma^{\vec{t}} : \widetilde{\mathcal{V}} \to \widetilde{\mathcal{V}}$ is the shift map defined by $\sigma^{\vec{t}}(\widetilde{\mathcal{V}}) = \widetilde{\mathcal{V}} + \vec{t}$. Thus, $\widetilde{\mathcal{V}}$ is *topologically transitive* (there exists an element whose orbit is dense).

Theorem 5.1.1 [Rud89] *If $\mathcal{V}$ has only a finite number of local patterns, then $(\widetilde{\mathcal{V}}, \sigma^{\vec{t}})$ is a compact metric space.*

Proof First, we show that $\widetilde{\mathcal{V}}$ is a complete metric space. Without loss of generality, we assume all tiles have diameter greater than $\frac{1}{2}$. Otherwise, we can expand the tiles by a factor of 2^l for sufficiently large $l \in \mathbf{N}$.

Suppose $\{\mathcal{V}_k\}$ is a Cauchy sequence in $\widetilde{\mathcal{V}}$. For each $i \geq 1$, there exists $n_i \in \mathbf{N}$ such that

$$m(\mathcal{V}_{k_1}, \mathcal{V}_{k_2}) < \frac{1}{2^i}$$

for $k_1, k_2 \geq n_i$. For each n_i, let P_{n_i} be the smallest patch of $\mathcal{V}_{n_i}$ which contains $B_{2^i}(0)$. By definition for each $i \in \mathbf{N}$ there exists a vector $\vec{t}_{n_i}$ with $\|\vec{t}_{n_i}\| \leq \frac{1}{2^i}$ such that $P_{n_i} + \vec{t}_{n_i}$ is a "sub-patch" of $P_{n_{i+1}}$ since $m(\mathcal{V}_{n_i}, \mathcal{V}_{n_j}) < \frac{1}{2^i}$. Let $\vec{t} = \sum_{i=1}^{\infty} \vec{t}_{n_i}$. Let tiling $\mathcal{V}'$ be the tiling that agrees with $P_{n_i} + (\vec{t} - \sum_{j=i}^{\infty} \vec{t}_{n_i})$ on $B_{2^i}(0)$. Tiling $\mathcal{V}'$ is well defined since

$$\|\vec{t}\| = \|\sum_{i=1}^{\infty} \vec{t}_{n_i}\| \leq \sum_{i=1}^{\infty} \|\vec{t}_{n_i}\| \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \leq 1 < \infty,$$

and since $P_{n_i} + \sum_{j=i}^{k-1} \vec{t}_{n_j}$ is a sub-patch of P_{n_k} for $1 \leq i < k$. The tiling $\mathcal{V}'$ is in $\widetilde{\mathcal{V}}$ since the patches of $\mathcal{V}'$ consist of the same tile types as in the tilings $\{\mathcal{V}_k\}$. This makes $\widetilde{\mathcal{V}}$ a complete metric space.

But $\widetilde{\mathcal{V}}$ is also a *totally bounded space* (for every $\epsilon > 0$ there exists a finite subset $F_\epsilon \subset \widetilde{\mathcal{V}}$ so that for each $\mathcal{V}_1 \in \widetilde{\mathcal{V}}$ there is some $\mathcal{V}_F \in F_\epsilon$ so that $m(\mathcal{V}_1, \mathcal{V}_F) < \epsilon$). This follows from the property that $\mathcal{V}$ has only a finite number of local patterns. Finally, every complete, totally bounded metric space is compact. $\square$

We can define cylinder sets for a tiling space $\widetilde{\mathcal{V}}$ by requiring tilings to have patches at specified places. A *cylinder set* for patch P and measurable set $A \subset \mathbf{R}^n$ is

$$U_{P,A} = \{\mathcal{V} : P + \vec{g} \text{ is a } \mathcal{V}\text{-patch for some } \vec{g} \in A\}. \tag{5.2}$$

5.2 Self-similar Tilings

Let $C : \mathbf{R}^n \to \mathbf{R}^n$ be an expansive linear map (i.e. all eigenvalues have modulus larger than 1). A tiling $\mathcal{V}$ is C*-subdividing* if:

1) for each $U \in \mathcal{V}$, CU is a union of tiles in $\mathcal{V}$.

2) $U = \vec{t_1} + U' \Longleftrightarrow CU = \vec{t_2} + CU'$ for some $\vec{t_1}, \vec{t_2} \in \mathbf{R}^n$.

A tiling is called *self-affine* with expansion (inflation) map C, if it is C-subdividing, has a

Figure 5.2: Subdivision of a self-similar Penrose tiling.

finite number of local patterns, and has the local isomorphism property. If C is a similarity, the tiling is called *self-similar*. A subdivision of a patch from a Penrose tiling is shown in Figure 5.2.

In our example, the tiles of $\mathbf{R}$ are translations of intervals with lengths h_1, h_2, h_3, h_4. These lengths are determined by the heights of the elements of the Markov partition $\mathcal{P}$. So, a tiling can also be thought as a mapping $\mathcal{U} : \mathbf{R} \to \{1, 2, 3, 4\}$ defined by

$$\mathcal{U}^{-1}(i) = \bigcup_k (h_i)_k$$

where $(h_i)_k = h_i + s_{k,i}$ for some $s_{k,i} \in \mathbf{R}$, for each $i = 1, 2, 3, 4$. We have $\mathbf{R} = \cup_k h_i + s_{k,i}$ where we assume this union is essentially disjoint (overlaps only at the endpoints). For our example, $\widetilde{\mathcal{U}}$ is the space of all such tilings of $\mathbf{R}$.

The tilings of interest are determined by the flow F^t of points in M whose trajectories do not contain any endpoints of the subintervals of J. Let $M^o = M \backslash \{(s, y) : F^t(s, y) \in \{0, \lambda_1, \lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \lambda_3\}$ for some $t \in \mathbf{R}\}$ be the surface M without the singular leaf of $\mathcal{F}$. We define the mapping $\tau : M^o \to \widetilde{\mathcal{U}}$ by

$$\tau(s, y)(t) = p(F^t(s, y))$$

where the mapping $p : M^o \to \{1, 2, 3, 4\}$ determines the partition element containing $F^t(s, y)$ at time t. Note that τ is 1:1 since the coding map ϕ defined in (3.2) is 1:1. Thus, we have $\tau(F^t(s, y)) = \sigma^t(\tau(s, y))$ for all $t \in \mathbf{R}$ and $(s, y) \in M^o$.

Since our flow F^t on M is not defined on the singular leaf, we revise the base of our surface from the interval J to the shift space X (two-sided) to obtain a flow without any singularities. The relationship between J and X was described by the map $\phi : J \to X$ in Theorem 3.2.1. Thus, we have defined a new "surface" Y with base X and height determined by $\widetilde{h}(x) = h_{x_0}$. Thus, points in Y are pairs (x, y) where $x \in X$ and $y \in [0, \widetilde{h}(x))$.

The corresponding vertical flow on Y, denoted $G^t : Y \to Y$, is non-singular because we have separated the singular leaf by doubling it into two separate leaves since

$card\{\rho^{-1}(s)\} = 2$ for $s \in J \backslash J^o$ (the set of pre-images of endpoints of the subintervals of J). Thus, every leaf is isomorphic to $\mathbf{R}$.

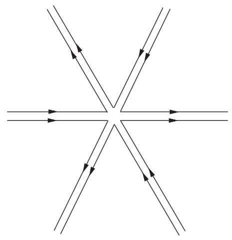

Figure 5.3: Neighborhood of the singularity after doubling the singular leaf.

We can also extend the map $\tau : M^o \to \widetilde{\mathcal{U}}$ to a map $\tau^* : Y \to \widetilde{\mathcal{U}}$ defined by $\tau^*(x,y)(t) = p^*\, G^t(x,y)$, where $p^*(x,y) = (\ ...\ h_{x_{-1}} \dot{h}_{x_0} h_{x_1}\ ...\ )$ and $\dot{h}_i$ designates the tile which contains the origin. Since τ^* is uniformly continuous on $\phi(J) \times \widetilde{h}(\phi(J))$, which is dense in Y, there exists a continuous extension to Y. Note $\tau^*(x,y)(t) = \sigma^t(\ ...\ h_{x_{-1}} \dot{h}_{x_0} h_{x_1} h_{x_2}\ ...\ )$ for $x = x_0 x_1 x_2 ... \in X$ and $y \in [0, h_{x_0})$.

For our example let the tiling dynamical system $\widetilde{\mathcal{V}}$ be defined by the orbit closure of any tiling $\mathcal{V} = \tau(s,y) = \tau^*(\phi(s), y)$ where the trajectory of $(s,y) \in M^o$ under the flow F^t does not contain any endpoints of the subintervals of J.

Theorem 5.2.1 *The tiling space $\widetilde{\mathcal{V}}$ is a self-similar tiling space with expansion map $C(y) = \theta y$.*

Proof The map $C : \mathbf{R} \to \mathbf{R}$ defined by $C(y) = \theta y$ is a similarity. The union of tiles in Ch_i is described by the substitution map S. Also, tiling $\mathcal{V}$ has only a finite number of local patterns since it is a tiling of $\mathbf{R}$. Tiling $\mathcal{V}$ also has the local isomorphism property since it is the suspension of a 1-dimensional substitution which is minimal [Que87]. □

Because the $\mathbf{R}$-action of σ^t on $\widetilde{\mathcal{V}}$ is continuous and $\widetilde{\mathcal{V}}$ is compact, there exists an invariant (Borel probability) measure μ on $\widetilde{\mathcal{V}}$ such that $\mu(A) = \mu(\sigma^t A)$ for all $t \geq 0$ and measurable sets $A \subset \widetilde{\mathcal{V}}$.

Recall the following definitions for $(\widetilde{\mathcal{V}}, \sigma^t, \mu)$. A function $f \in L^2(\widetilde{\mathcal{V}}, \sigma^t, \mu)$ is an *eigenfunction* and $\alpha \in \mathbf{R}$ is its corresponding *eigenvalue* for the $\mathbf{R}$-action on $\widetilde{\mathcal{V}}$. This means that $f \not\equiv 0$ and

$$f(\sigma^t \mathcal{V}) = e^{2\pi i \alpha t} f(\mathcal{V}), \quad \text{for all } \ t \in \mathbf{R}, \ \mathcal{V} \in \widetilde{\mathcal{V}}.$$

The map σ^t is *ergodic* if $\sigma^t(A) = A$ for some $t > 0$ and measurable set $A \subset \widetilde{\mathcal{V}}$ then $\mu(A) = 0$ or $\mu(A) = 1$. The map σ^t is *weakly mixing* if there are no non-constant eigenfunctions. The map σ^t is *(strongly) mixing* if for every $t \in \mathbf{R}$ and measurable set $A \subset \widetilde{\mathcal{V}}$, we have

$$\lim_{n\to\infty} \mu[A \cap \sigma^{C^n t} A] \to [\mu(A)]^2$$

as $n \to \infty$.

Theorem 5.2.2 *The* $\mathbf{R}$*-action on the tiling space* $\widetilde{\mathcal{V}}$ *is not (strongly) mixing.*

Strongly mixing fails since the tiling space is self-affine [Sol97].

Theorem 5.2.3 *The* $\mathbf{R}$*-action on the tiling space* $\widetilde{\mathcal{V}}$ *is weakly mixing.*

Proof This follows from [Sol97] since θ is not Pisot. For completeness we give a proof here. We show the $\mathbf{R}$-action on $\widetilde{\mathcal{V}}$ is weakly mixing by proving there are no non-constant eigenfunctions.

Suppose the contrary that $f \in L^2(\widetilde{\mathcal{V}}, \mu)$ is an eigenfunction and $\alpha \in \mathbf{R}$ is its corresponding eigenvalue for the $\mathbf{R}$-action on $\widetilde{\mathcal{V}}$. We can assume $|f| = 1$ almost everywhere since $\widetilde{\mathcal{V}}$ is topologically transitive.

Let $\epsilon > 0$. By Lusin's Theorem [Dud89], there exist a set $E \subset \widetilde{\mathcal{V}}$ such that f is continuous on E and $\mu(E) > 1 - \epsilon$.

For each $n \in \mathbf{N}$, define the patch $A_n = (\dot{h}_{u_0} h_{u_1} ... h_{u_n})$ to be the $n+1$ tiles associated with u about the origin. We claim for every A_n there exist $t_n \in \mathbf{R}$ such that $\mu(\sigma^{t_n} U_{A_n,0} \cap E) > (1-\epsilon)\mu(U_{A_n,0})$. Suppose the contrary that for some A_n we have $\mu(\sigma^s U_{A_n,0} \cap E^c) > \epsilon\mu(U_{A_n,0})$ for every $s \in \mathbf{R}$. Let $\{s_k\}$ be a sequence of numbers such that $\{\sigma^{s_k} U_{A_n,0} \cap E\} = \{U_{A_n,s_k} \cap E\}$ is a disjoint sequence of sets. Then, $\mu[\cup_{k=1}^{\infty}(\sigma^{s_k} U_{A_n,0} \cap E^c)] = \sum_{k=1}^{\infty} \mu(\sigma^{s_k} U_{A_n,0} \cap E^c) > \mu(E^c)$ which is a contradiction.

Since f is continuous on E, there exists an $N > 0$ such that for each $n \geq N$, we have $\mathcal{W}_1, \mathcal{W}_2 \in U_{\sigma^{t_n} A_n, 0} = U_{A_n, t_n} \Rightarrow m(\mathcal{W}_1, \mathcal{W}_2) < \epsilon$. But we also claim that if $\mathcal{W} \in U_{A_{|S^{N+2}(1)|}, t_n} \subset U_{A_{|S^{N}(1)|}, t_n}$, then $\sigma^{\theta^{N+1} h_1}(\mathcal{W}) \in U_{A_{|S^{N}(1)|}, t_n}$. This follows from an equivalent statement for substitutions, namely

$$\sigma^{|S^{n+1}(1)|} u = [S^n(1)] u_{|S^n(1)|+1} u_{|S^n(1)|+2} \cdots$$

which can be proved by induction.

Since μ is a σ^t-invariant measure, $\mu(A_{N+2}, t_N) = \mu(\sigma^{\theta^N h_1} A_{N+2}, t_N)$, so we can choose a tiling $\mathcal{W} \in U_{A_{N+2}, t_N} \subset U_{A_N, t_N}$ such that $\sigma^{\theta^N h_1} \mathcal{W} \in U_{A_N, t_N}$. Thus, we have

$$\begin{aligned} |f(\mathcal{W}) - f(\sigma^{\theta^n h_1} \mathcal{W})| &< \epsilon, \;\; n \geq N \\ |f(\mathcal{W}) - (e^{2\pi i \alpha})^{\theta^n h_1} f(\mathcal{W})| &< \epsilon, \;\; n \geq N \;\text{ by (5.3)} \\ |f(\mathcal{W})[(e^{2\pi i \alpha})^{\theta^n h_1} - 1]| &< \epsilon, \;\; n \geq N. \end{aligned}$$

Thus, $\alpha\theta^n h_1 \to 0(mod\mathbf{Z})$. So, $\gamma\theta^n \to 0(mod\mathbf{Z})$ where $\gamma = \alpha h_1$. This is equivalent to θ being a Pisot number since θ is algebraic [ST87], which is a contradiction. □

Since $\widetilde{\mathcal{V}}$ is weakly mixing and not strongly mixing with respect to the $\mathbf{R}$-action, the flow G^t is also weakly mixing and not mixing on Y. Since G^t is an almost 1:1 extension of the flow F^t via $(x, y) \to (\rho(x), y)$, F^t is also weakly mixing and not strongly mixing on M^o.

Theorem 5.2.4 *The flow $F^t : M^o \to M^o$ is minimal, uniquely ergodic, weakly mixing but not (strongly) mixing.*

This follows from Lemma 2.4.3, Theorem 3.2.2, Theorem 5.2.2, and Thereom 5.2.3.

5.3 2-dimensional Self-similar Tilings

In this section we construct a 2-dimensional tiling dynamical system. For two points in M, we can define two 1-dimensional tilings of $\mathbf{R}$ as described in the last section. By taking the cross product of two 1-dimensional tilings, we obtain a 2-dimensional tiling of $\mathbf{R}^2$. Thus, the tiles are rectangles with dimensions $h_i \times h_j$. Let $\widetilde{\mathcal{U}}_2$ be the space of all such tilings of $\mathbf{R}^2$.

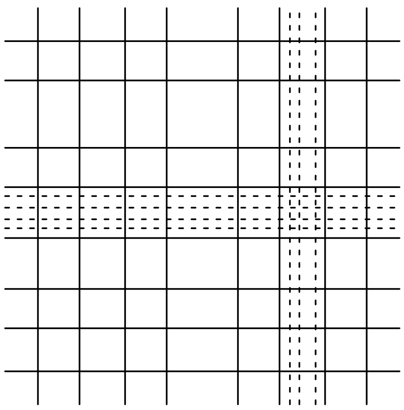

Figure 5.4: 2-dimensional patch of $\mathcal{U}_2$ with one subdivision of a tile.

We define the mapping $\tau_2 : M^o \times M^o \to \widetilde{\mathcal{U}}_2$ by

$$\tau_2(z_1, z_2)(\vec{t}) = (p(F^{t_1} z_1), p(F^{t_2} z_2)),$$

where $z_1, z_2 \in M^o$ and $\vec{t} = (t_1, t_2) \in \mathbf{R}^2$. Let the mapping $(\sigma_2)^{\vec{t}} : \widetilde{\mathcal{U}}_2 \to \widetilde{\mathcal{U}}_2$ be the translation by $\vec{t} \in \mathbf{R}^2$ which is defined by

$$(\sigma_2)^{\vec{t}}\, \mathcal{U}_2 = \mathcal{U}_2 + \vec{t},$$

where $\vec{t} \in \mathbf{R}^2$ and $\mathcal{U}_2 \in \widetilde{\mathcal{U}}_2$. Similar to last section, we have $\tau_2(F^{t_1} z_1, F^{t_2} z_2) = (\sigma_2)^{\vec{t}}(\tau_2(z_1, z_2))$ where $\vec{t} = (t_1, t_2) \in \mathbf{R}^2$.

Theorem 5.3.1 *The tiling space $\widetilde{\mathcal{U}}_2$ is a self-affine tiling space with expansion map $C : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $C(z) = \theta z$.*

Theorem 5.3.2 *The $\mathbf{R}^2$-action on the tiling space $\widetilde{\mathcal{U}}_2$ is weakly mixing but not (strongly) mixing.*

Proof Strongly mixing fails since the tiling space is self-affine. Weakly mixing follows since the expansion coefficient θ is non-Pisot [Sol97]. $\square$

Chapter 6

Beta-transformations

In this chapter, we generalize the notion of β-transformations to be defined in terms of self-inducing interval exchange transformations.

6.1 Definitions

For any real number $\beta > 1$, the *β-transformation* $T_\beta : [0,1) \to [0,1)$ is defined by

$$T_\beta x = \beta x (\text{mod } 1)$$

for $x \in [0,1)$. If $\beta \in \mathbf{Z}$, then the Lebesgue measure is invariant for T_β. So, the more interesting case is for $\beta \notin \mathbf{Z}$. A. Rényi [R57] showed there exists a T_β-invariant measure μ_β that is *equivalent* to the Lebesgue measure ($\mu_\beta(U) = \int_U h_\beta(x)dx$, for some function $h_\beta(x) > 0$). He also proved that T_β is ergodic for $\beta \notin \mathbf{Z}$. Later, W. Parry [Par60] explicitly determined the invariant measure μ_β. Parry also showed that T_β is weakly mixing for $\beta \notin \mathbf{Z}$. The β-transformation T_β for $\beta = \theta$, is graphed in Figure 6.1. Recall $\theta > 1$ is the largest root of the characteristic polynomial $P(x) = x^4 - 7x^3 + 13x^2 - 7x + 1$ of matrix A defined in Section 2.4.

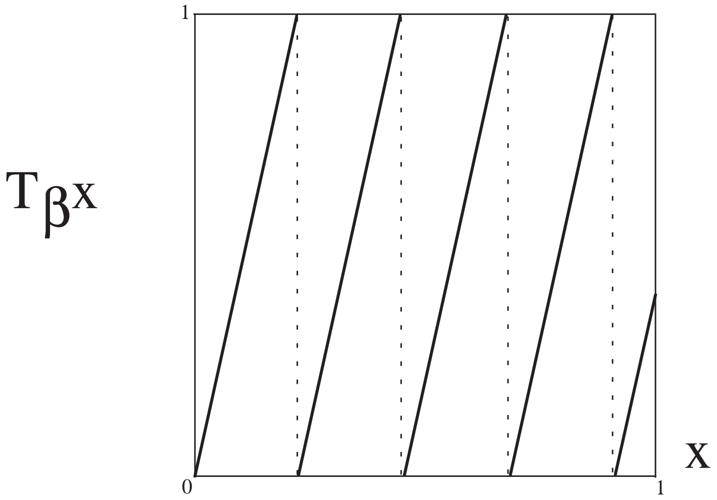

Figure 6.1: β-transformation $T_\beta : [0,1) \to [0,1)$.

6.2 β-expansions

β-transformations are associated to β-expansions that were first studied by A. Rényi [Ŕ57]. For each $x \in [0,1]$, there exists a β-*expansion* of x, denoted $d(x,\beta) = \{d_i\}$, that is a sequence in $\{0,1,2,...,\lfloor\beta\rfloor\}^{\mathbf{N}}$ where $\lfloor\beta\rfloor$ is the greatest integer strictly less than β. The β-expansion of a number $x \in [0,1]$ is calculated by considering the iterates of the map $E : \mathbf{R} \to \mathbf{R}$ that is defined by

$$E(x) = \beta x - \lfloor\beta x\rfloor.$$

Then, $d_i = \lfloor\beta E^{i-1}(x)\rfloor$ and we can write x in base β by

$$x = d_1\beta^{-1} + d_2\beta^{-2} + \ \ldots \ + d_k\beta^{-k} + \ \ldots$$

where $0 \le d_i \le \lfloor\beta\rfloor$.

Example 1. [Thu89] Let β be the largest (Pisot) root of $x^2 - x - 1$. Then, $\beta \approx 1.618$ and $d(1,\beta) = .10101010....$

Example 2. [Thu89] Let β be the largest (non-Pisot) root of $x^{11}-3x^{10}-2x^9-x^8-2x^7-3x^6-x^4-2x^3-3x^2-x-1$. Then, $\beta \approx 3.67558944$ and $d(1,\beta) = .321230123103212301231\!0....$

Theorem 6.2.1 [Par60] *A sequence of positive integers $\{d_i\}$ is a β-expansion if and only if the sequence has an infinite number of $d_i \neq 0$, and no sequence obtained by a finite number of shifts is lexicographically greater than it.*

Theorem 6.2.2 [Par60] *If β is a Pisot number, then the β-expansion of 1 is eventually periodic (with respect to the shift map).*

The converse of this Theorem is not true, as shown by Example 2. Unfortunately, θ is not a counterexample to the converse since $d(1,\theta) = .41302212230002113002041....$ is not eventually periodic since there is another real root of the characteristic polynomial [Bla89].

Theorem 6.2.3 [Thu89] *For any eventually periodic β-expansion of 1, there exists a self-similar tiling of* $\mathbf{R}$.

Proof A *β-integer* is a nonnegative real number whose β-expansion has no decimals and is lexicographically less than the β-expansion of 1, after appropriate shifts to equate the number of digits to the left of the decimal. The self-similar tiling of $\mathbf{R}$ is defined by tiles that are the intervals between consecutive β-integers.

6.3 Generalized β-transformations

Given an interval exchange transformation that induces itself, we construct a *generalized β-transformation* $G_\beta : J \to J$ in the following way. Suppose $T_{\vec{\lambda},\pi} : J \to J$ self-induces on $J' \subset J$ where $J' = \frac{1}{\beta}J$. First, we build the Kakutani skyscraper above J' by applying $T_1 : J \to J' \times \mathbf{N}$ by $T_1(s) = (s', n)$ where $s' = T^{-n}s$ for some $s' \in J'$. Next, we stretch the base by a factor of β, which stretches the base to the interval J. This amounts to applying the map $T_2 : J' \times \mathbf{N} \to J \times \mathbf{N}$ where $T_2(s, y) = (\beta s, y)$. Finally, we collapse the tower onto the base interval J by applying the map $T_3 : J \times \mathbf{N} \to J$ defined by $T_3(s, y) = s$. Thus, $G_\beta = T_3 \circ T_2 \circ T_1 : J \to J$. Figure 6.2 shows the generalized β-transformation G_β associated with $T_{\vec{\lambda},\pi}$, defined in Section 2.4, where $\beta = \theta$ and $\{\alpha_0, \alpha_1, ..., \alpha_{11}\}$ defines the partition Q in Section 4.1. The G_β-invariant measure is $\mu(I) = h_1\lambda(I \cap J_1) + h_2\lambda(I \cap J_2) + h_3\lambda(I \cap J_3) + h_4\lambda(I \cap J_4)$ where λ is the Lebesque measure.

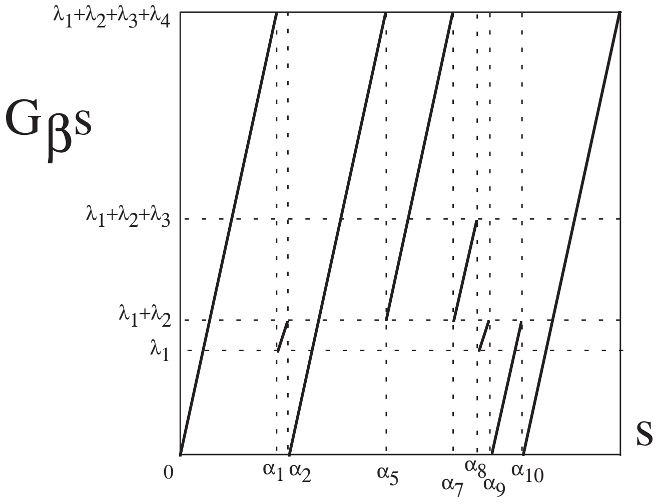

Figure 6.2: Generalized β-transformation $G_\beta : J \to J$.

6.4 Natural extensions

Natural extensions are the smallest invertible extension of a non-invertible map. In order to define an extension that is 1:1, we must define "past" images in a unique way such that their extensions agree with the "future" images already defined by the non-invertible map. K. Dajani and C. Kraaikamp [DK89] explicitly construct natural extensions for β-transformations. A map $(Y, \mathcal{C}, \nu, \bar{g})$ is a *natural extension* [DK89] of a map $(X, \mathcal{B}, \mu, g)$ if g

is non-invertible and $\bar{g}$ is invertible and there exists a surjective map $\psi : Y \to X$ such that

(i) $\psi^{-1}\mathcal{B} \subseteq \mathcal{C}$,

(ii) $g \circ \psi = \psi \circ \bar{g}$,

(iii) $\nu(\psi^{-1}(U)) = \mu(U)$ for all $U \in \mathcal{B}$, and

(iv) $\vee_{m=0}^{\infty}\bar{g}^m\psi^{-1}\mathcal{B} = \mathcal{C}$ where $\vee_{m=0}^{\infty}\bar{g}^m\psi^{-1}\mathcal{B}$ is the smallest σ-algebra containing $\bar{g}^m\psi^{-1}\mathcal{B}$.

Theorem 6.4.1 *The self-inducing interval exchange transformation $T_{\vec{\lambda},\pi}$ induces a generalized β-transformation G_β, for $\beta = \theta$, whose natural extension is the pseudo-Anosov diffeomorphism D^{-1}, defined in Section 4.1.*

Proof We need to show $(M, \mathcal{B}_2, \lambda, D)$ is the natural extension for $(J, \mathcal{B}_1, \mu, G_\theta)$ where $\mathcal{B}_2$ is the Borel σ-algebra on the surface M with Lebesque measure λ and $\mathcal{B}_1$ is the Borel σ-algebra on the interval J with invariant measure μ.

Let $\psi : M \to J$ be defined by $\psi(s, y) = s$. So, subintervals of J are mapped by ψ^{-1} to rectangular sets of heights h_i in M. This proves (i). Condition (iv) is true since we can make rectangular sets, generated by $\psi^{-1}(s_1, s_2)$, as narrow as required by changing the width of the interval $(s_1, s_2) \subset J$. The height of these rectangles can be as short as needed by applying D^{-1}, which contracts the heights of sets in M by a factor of $\frac{1}{\theta}$.

We will prove (ii) for $(s, y) \in Q_2$. So, we have $s \in [\alpha_1, \alpha_2) = [\lambda_1, -\lambda_1 + \lambda_4)$ and $y \in [0, h_2)$.

$$G_\theta \circ \psi(s, y) = G_\theta(s) = \theta(s - \lambda_1) + \lambda_1$$
$$\psi \circ D^{-1}(s, y) = \psi[\theta(s - \lambda_1) + \lambda_1, \frac{1}{\theta}y] = \theta(s - \lambda_1) + \lambda_1$$

A similar argument is used for (s, y) in other partition elements of $\mathcal{Q}$.

Also, let $(s_1, s_2) \subset J_i$ for some $i = 1, 2, 3, 4$. Then,

$$\lambda(\psi^{-1}(s_1, s_2)) = \lambda[(s_1, s_2) \times [0, h_i)] = h_i(s_2 - s_1) = \mu(s_1, s_2).$$

This proves condition (iii). □

Chapter 7

Conclusions

One aspect of this dissertation considers the need for Pisot numbers. The use of Pisot numbers is not always necessary for certain examples in the subjects discussed here. In particular, a non-Pisot expansion coefficient for a pseudo-Anosov diffeomorphism is not only possible as shown here, but can be rewarded when applying the coefficient to be the expansion coefficient of a self-similar tiling. Also, it is not necessary that numbers $\beta > 1$ be Pisot numbers to have eventually periodic β-expansions of 1.

A closed path of length 8 in graph G was necessary for a non-Pisot, self-induced interval exchange transformation. All closed paths of length less than 8 have Rauzy induction matrices with an eigenvalue that is Pisot. Also, each closed path considered here, whose Rauzy induction matrix has a non-Pisot eigenvalue, consisted of at least two closed paths. One closed paths traverses the top of G through permutation $(143)(2)$ and another traverses the bottom of G through permutation $(134)(2)$. This observation seems too regular to be just a coincidence.

Many of the characteristic polynomials of the Rauzy induction matrices, including the example studied here, satisfy the conditions for an Alexander polynomial of knots, $\triangle(t) = \triangle(t^{-1})$ after normalization and $\triangle(1) = 1$. Using knot theory, a surface can be constructed from a knot, which is determined by its Alexander polynomial, that is viewed in $\mathbf{R}^3$. I would be interested to see if the surface constructed using this method is the same surface in this dissertation.

Since the dissertation is concerned with induction of interval exchange transformations, I did not seek other ergodic properties of T. However, I would conjecture that T has many ergodic properties, including weakly mixing.

Concerning tiling dynamical systems, many questions are left unanswered about 1-dimension projections of the 2-dimension system described in Chapter 5. There are also many other ergodic properties that are yet undetermined, such as topologically weak mixing.

Appendix A

Mathematica Programs

In this appendix, we give the Mathematica programs created to search for an interval exchange transformation with a "non-Pisot self-induction". Many properties of the transformation $T_{\vec{\lambda},\pi}$ are checked by the programs in order to produce the example given here. However, the programs are written to accept any general interval exchange transformation. For example, the programs accept $\vec{\lambda}$ to be any positive vector in $\mathbf{R}^m$ and any permutation $\pi \in S_m$. The Mathematica version used is 2.2.

A.1 Rauzy Induction

A.1.1 ProgramT

The module `ProgramT` calculates $T_{\vec{\lambda},\pi}(x)$, given $\vec{\lambda}$ and π, and plots a graph using a subroutine called `EvaluateT`.

```
ProgramT[perm_List,lambda_List] :=
Module[{piepairs,xpie,xpieinverse,xlambdapie,linepairs},

m=Length[lambda];
beta[0] = 0;
Do[beta[i]=beta[i-1]+lambda[[i]],{i,1,m}];

piepairs = {};
ctop[cycle_List] :=
```

```
piepairs=Union[piepairs,Transpose[{cycle,RotateLeft[cycle]}]];
funct[{x_Integer,y_Integer}] := pie[x]=y;
Map[ctop,perm];
Do[If[Position[piepairs,i]=={},piepairs=Append[piepairs,{i,i}],Null],{i,m}];
Map[funct,piepairs];

inversefunction[g_,i_Integer,m_Integer] :=
NestList[g,i,m][[Position[NestList[g,i,m],i][[2,1]]-1]];
Do[pieinverse[i] = inversefunction[pie,i,m],{i,m}];

lambdapie = Table[lambda[[pieinverse[i]]],{i,1,m}];

bettapie[0] = 0;
Do[bettapie[i]=bettapie[i-1]+lambdapie[[i]],{i,1,m}];

EvaluateT[x_] :=
Do[If[betta[i-1] <= x < betta[i],
T[x]=x+bettapie[pie[i]-1]-betta[i-1]], {i,1,m}];

Do[EvaluateT[betta[i-1]];
EvaluateT[betta[i]-.001],
{i,1,m}];
linepairs=Flatten[Table[{{betta[i-1],T[betta[i-1]]},
           {betta[i],T[betta[i]-.001]}},{i,1,m}],1];
Show[Graphics[Line[linepairs]],Axes->True,AxesOrigin->{0,0}]

]
```

A.1.2 NCMMatrices

The module `NCMVectors` uses the Mathematica command

NonCommutativeMultiply$[x_1, x_2] = x_1 ** x_2$ to multiply (dot product) vectors in a non-commutative way. `NCMVectors` also removes any terms that are 0 since Mathematica does not recognize that $0 ** d = 0$ or $d ** 0 = 0$. The vector $\vec{w} = \vec{u} ** \vec{v}$ is the resulting vector.

```
NCMVectors[u_List,v_List] :=

Module[{ridpositions,tempu,tempv,temp},
```

```
ridpositions = Join[Position[u,0],Position[v,0]];
tempu = Delete[u,ridpositions];
tempv = Delete[v,ridpositions];
temp = MapThread[NonCommutativeMultiply,{tempu,tempv}];
temp = Map[Distribute,temp];
If[temp == {}, w = 0, w = Apply[Plus,temp]]

]
```

The module `NCMMatrices` uses `NCMVectors` to multiply two matrices. The matrix $S3 = S1 ** S2$ is the resulting matrix, which gives all paths in graph G of a certain length.

```
NCMMatrices[S1_List,S2_List] :=

Module[{m,n,temp},

m = Dimensions[S1][[1]];
n = Dimensions[S2][[2]];
S3 = Table[0,{i,m},{j,n}];

If[Dimensions[S1][[2]] == Dimensions[S2][[1]],
Do[(NCMVectors[S1[[i]],Transpose[S2][[j]]];S3[[i,j]]=w),{i,m},{j,n}],
Print["Incorrect dimensions"]];
Print["The resulting matrix is S3"]

]
```

A.1.3 PossiblePaths

The module `PossiblePaths` removes the characters "*" and "_" from the results of noncommutative multiplication. It also replaces "+" with a comma since "+" separates distinct paths. The resulting output is *paths*, which lists all paths of a certain length.

```
PossiblePaths[ncmentry_] :=

Module[{n,templist,xpaths},

templist = Characters[ToString[ncmentry]];
n = Length[Position[templist, "*"]] / (2 * Length[Position[templist, "+"]]
```

```
+ 2) + 1;

templist = Delete[templist,Position[templist,"*"]];
templist = Delete[templist,Position[templist," "]];
templist = Delete[templist,Position[templist,"+"]];

templist = Partition[templist,n];
paths = Map[StringJoin,templist];
Print["Paths of length ",n," are ",paths]

]
```

The module `Step` calculates the induced permutation $c_a(\pi)$ or $c_b(\pi)$ and Rauzy induction matrix $A_a(\pi)$ or $A_b(\pi)$ for one induction step from the permutation π. The letter c is either a or b, which designates an a-induction or b-induction.

```
Step[perm_List,c_String] :=

Module[{tempmat,xfinalmat,piepairs,pie,pieinverse,cpie,xcperm,r},

Clear[tempmat,finalmat,pie,pieinverse,cpie,cperm,r];
m = Max[perm];
If[ValueQ[A],Null,A=IdentityMatrix[m]];
piepairs = Table[{i,perm[[i]]}, {i,m}];
funct[{x_Integer,y_Integer}] := pie[x]=y;
Map[ctop,perm];
Do[If[Position[piepairs,i]=={},piepairs=Append[piepairs,{i,i}],Null],{i,m}];
Map[funct,piepairs];

inversefunction[g_,i_Integer,m_Integer] :=
NestList[g,i,m][[Position[NestList[g,i,m],i][[2,1]]-1]];
Do[pieinverse[i] = inversefunction[pie,i,m],{i,m}];

If[c == "a",

(
cpie[i_] := Which[
1<=i<=pieinverse[m],pie[i],
i==pieinverse[m]+1,pie[m],
pieinverse[m]+2<=i,pie[i-1]];
```

```
If[ValueQ[path],path=StringInsert[path,"a",StringLength[path]+1],path="a"];
cperm=Table[cpie[i], {i,m}];

tempmat = IdentityMatrix[pieinverse[m]];
Do[r[i]=Join[tempmat[[i]],Table[0,{m-pieinverse[m]}]], {i,pieinverse[m]-1}];
r[pieinverse[m]]=Join[tempmat[[pieinverse[m]]],{1},
                  Table[0,{m-pieinverse[m]-1}]];

tempmat=Table[Switch[i-j,-1,1,_,0],{i,m-pieinverse[m]-1},{j,m-pieinverse[m]}];
Do[ r[i]=Join[Table[0,{pieinverse[m]}],tempmat[[i-pieinverse[m]]]],
{i,pieinverse[m]+1,m-1}];
r[m] = Join[Table[0,{pieinverse[m]}],{1},Table[0,{m-pieinverse[m]-1}]]
),

(
cpie[i_] := Which[
1 <= pie[i] <= pie[m], pie[i],
pie[m] < pie[i] < m, pie[i] + 1,
pie[i] == m, pie[m] + 1];

If[ValueQ[path],path=StringInsert[path,"b",StringLength[path]+1],path="b"];
cperm=Table[cpie[i], {i,m}];

tempmat = IdentityMatrix[m-1];
Do[r[i]=Join[tempmat[[i]],{0}],{i,m-1}];
r[m]=Join[Table[0,{pieinverse[m]-1}],{1},Table[0,{m-pieinverse[m]-1}],{1}]
)];

finalmat = Table[r[i],{i,m}];
A = A . finalmat;
Print["cperm=",cperm,"   A(pie,c)=",MatrixForm[finalmat],
" A=",MatrixForm[A]," path = ",path]

]
```

The module `Path` calculates the (product) Rauzy induction matrix A, defined by a given string of a's and b's using module `Step`. Since the search is for matrices whose characteristic polynomial has a non-Pisot number for a root, the module `Path` graphs the eigenvalues of A.

```
Path[perm_List,trail_String] :=

Module[{n,nextletter,nextperm,dummy,temptrail, xvals, eigencoords},

temptrail = trail;
nextperm = perm;
n = StringLength[temptrail];
Clear[A,path];

Do[(nextletter = StringTake[temptrail,{1}];
temptrail = StringDrop[temptrail,1];
Step[nextperm,nextletter];
nextperm=cperm), {i,n}];

vals = Eigenvalues[Transpose[A]]
eigencoords = {PointSize[0.03]};
Do[AppendTo[eigencoords,Point[{Re[vals[[i]]],Im[vals[[i]]]}]],{i,Length[vals]}];
AppendTo[eigencoords,Circle[{0,0},1]];

Print["Path = ", path, "      A = ",MatrixForm[A]];

If[Length[eigencoords] == 2, Print["No eigenvalues for A"],
Show[Graphics[eigencoords], Axes -> True]]

]
```

A.1.4 MassiveChecking

The module `MassiveChecking` prints all non-Pisot examples of paths of length m that begin (and end) with the permutation $\pi = (14)(23)$.

```
MassiveChecking[p_Integer] :=

Module[{temporaryS,npcounter},

<< S.m;
<< NCM.m;
<< PossiblePaths.m;
<< StepPath.m;

Print["paths length   paths     non-Pisot paths"];
```

```
Print["------------   -----     ---------------"];
temporaryS = S;
Do[
  npcounter = 0;
  NCMMatrices[S,temporaryS];
  temporaryS = S3;
  PossiblePaths[temporaryS[[4,4]]];
  If[paths[[1]] != "0",
   Do[
     Path[{4,3,2,1},ToString[paths[[k]]]];
     vals = Sort[Abs[N[vals]]];
     If[vals[[3]] > 1, npcounter=npcounter+1, Null],
   {k,1,Length[paths]} ],
   Null];
  Print["     ",i,"           ",Length[paths],"           ",npcounter],
{i,2,p}]

]
```

A.2 Surfaces Built over Transformations

The next step is to construct a surface based on the transformation $T_{\vec{\lambda},\pi}$. This amounts to defining a suspension space above the transformation using two positive vectors $\vec{h}$, $\vec{a} \in \mathbf{R}^m$. These vectors are the Perron-Frobenius eigenvectors of $A^t\vec{h} = \theta\vec{h}$ and $A\vec{\lambda} = \theta\vec{\lambda}$ with Perron-Frobenius eigenvalue $\theta > 1$. Note we must have $A^k > 0$ for some $k > 0$ to apply the Perron-Frobenius Theorem. The vector $\vec{a}$ is defined by a matrix equation of the type $L\vec{h} + \vec{a} = \theta\vec{a}$ where L is a positive integer matrix .

A.2.1 Diagram

The module `Diagram` draws the surface in rectangular form with the side identified by colors, given the vectors $\vec{\lambda}, \vec{h}, \vec{a}$, and the permutation π.

```
Diagram[lambda_List,inh_List,ina_List,perm_List] :=
```

```
Module[{inc,piepairs,xpie,xsigma,xpieinverse,lambdapie},

Clear[betta,piepairs,a,h,pie,pieinverse,sigmabettapie];
m=Length[lambda];
betta[0] = 0;
Do[betta[i]=betta[i-1]+lambda[[i]],{i,1,m}];
inc = betta[m]/100;

piepairs = Table[{i,perm[[i]]}, {i,m}];
funct[{x_Integer,y_Integer}] := pie[x]=y;
Map[ctop,perm];
Map[funct,piepairs];

inversefunction[g_,i_Integer,m_Integer] :=
NestList[g,i,m][[Position[NestList[g,i,m],i][[2,1]]-1]];
Do[pieinverse[i] = inversefunction[pie,i,m],{i,m}];

a = ina;
h = inh;
AppendTo[a,0];
AppendTo[h,0];
h[[0]] = 0;
a[[0]] = 0;

sigma[j_] := Which[j==0, pieinverse[1]-1, j==pieinverse[m], m,
1 <= j <= m && j != pieinverse[m],pieinverse[pie[j]+1]-1];

lambdapie = Table[lambda[[pieinverse[i]]], {i,m}];

bettapie[0] = 0;
Do[bettapie[i]=bettapie[i-1]+lambdapie[[i]],{i,1,m}];

horzline = Table[
{RGBColor[Mod[i,2], Mod[Quotient[i,2],2], Mod[Quotient[i,4],2] ],
Line[{{betta[i-1],h[[i]]},{betta[i],h[[i]]}}],
Line[{{bettapie[pie[i]-1],0},{bettapie[pie[i]],0}}]},
{i,m}];

vertline = Table[
If[i == pieinverse[m],

If[a[[m]] >= 0,
```

```
{RGBColor[Mod[i,2]/1.8,Mod[Quotient[i,2],2]/1.8,Mod[Quotient[i,4],2]/1.8],
Line[{{betta[m],0},{betta[m],a[[m]]}}],
Line[{{betta[i],h[[i]]},{betta[i],a[[i]]}}]},

(* else: a[[m]] < 0 *)
{RGBColor[Mod[i,2]/1.8,Mod[Quotient[i,2],2]/1.8,Mod[Quotient[i,4],2]/1.8],
Line[{{betta[sigma[m]],a[[sigma[m]]]},{betta[sigma[m]],
        h[[sigma[m]+1]]-h[[m]]}}],
Line[{{betta[i]-inc,a[[i]]},{betta[i]-inc,h[[i]]}}]} ],

(* else: i != pieinverse[m] *)
{RGBColor[Mod[i,2]/2,Mod[Quotient[i,2],2]/2,Mod[Quotient[i,4],2]/2],
Line[{{betta[i]-inc,a[[i]]},{betta[i]-inc,h[[i]]}}],
Line[{{betta[sigma[i]],a[[sigma[i]]]},{betta[sigma[i]],
        h[[sigma[i]+1]]}}]}],{i,m-1}];

If[a[[m]] >= 0,
AppendTo[vertline,{RGBColor[Mod[m,2]/1.5,Mod[Quotient[m,2],2]/1.5,
Mod[Quotient[m,4],2]/1.5], Line[{{betta[m],a[[m]]},{betta[m],h[[m]]}}],
Line[{{betta[sigma[m]],a[[sigma[m]]]},{betta[sigma[m]],h[[sigma[m]+1]]}}]}],
AppendTo[vertline,{RGBColor[Mod[m,2]/1.5,Mod[Quotient[m,2],2]/1.5,
Mod[Quotient[m,4],2]/1.5], Line[{{betta[m],0},{betta[m],h[[m]]}}],
Line[{{betta[sigma[m]],h[[sigma[m]+1]]-h[[m]]},{betta[sigma[m]],h[[sigma[m]+1]]}
}]}]
];

Show[Graphics[Join[vertline,horzline]]]

]
```

A.3 Generalized β-expansions

A.3.1 Betaexp

The module `Betaexp` calculates the β-expansion for 1 where $\beta > 1$.

```
Betaexp[beta_Real,digits_Integer] :=

Module[{x,expansion,stringexpansion},

x=1;
```

```
expansion={};
Do[(x=beta*x;
    AppendTo[expansion,Floor[x]];
    x=x-Floor[x]),
{i,1,digits}];
stringexpansion={};
Do[AppendTo[stringexpansion,ToString[expansion[[i]]]],{i,1,Length[expansion]}];
stringexpansion=StringJoin[stringexpansion];
Print["carry(beta) = .",stringexpansion]

]
```

Appendix B

Exact Values

The non-Pisot example has the following parameter values.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 3 & 2 & 2 \end{pmatrix}$$

$$\theta = \frac{7 + \sqrt{5} + \sqrt{2}\sqrt{19 + 7\sqrt{5}}}{4}.$$

$$\pi = (14)(23).$$

$\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ where

$$\begin{aligned}
\lambda_1 &= -\frac{1}{4} - \frac{\sqrt{5}}{4} - \frac{\sqrt{2}\sqrt{19+7\sqrt{5}}}{8} + \frac{\sqrt{10}\sqrt{19+7\sqrt{5}}}{8}, \\
\lambda_2 &= \frac{5}{4} - \frac{\sqrt{5}}{4} + \frac{\sqrt{2}\sqrt{19+7\sqrt{5}}}{2} - \frac{\sqrt{10}\sqrt{19+7\sqrt{5}}}{4}, \\
\lambda_3 &= -\frac{7}{4} + \frac{3\sqrt{5}}{4} - \frac{\sqrt{2}\sqrt{19+7\sqrt{5}}}{2} + \frac{\sqrt{10}\sqrt{19+7\sqrt{5}}}{4}, \\
\lambda_4 &= 1
\end{aligned}$$

$\vec{h} = (h_1, h_2, h_3, h_4)$ where

$$
\begin{aligned}
h_1 &= -\frac{3}{4} + \frac{\sqrt{5}}{4} + \frac{\sqrt{2}\sqrt{19+7\sqrt{5}}}{8} + \frac{\sqrt{\frac{5}{2}}\sqrt{19+7\sqrt{5}}}{4}, \\
h_2 &= -\frac{5}{4} + \frac{3\sqrt{5}}{4} - \frac{\sqrt{2}\sqrt{19+7\sqrt{5}}}{8} + \frac{\sqrt{\frac{5}{2}}\sqrt{19+7\sqrt{5}}}{4}, \\
h_3 &= \frac{1}{2} + \frac{3\sqrt{2}\sqrt{19+7\sqrt{5}}}{8} - \frac{\sqrt{\frac{5}{2}}\sqrt{19+7\sqrt{5}}}{4}, \\
h_4 &= 1
\end{aligned}
$$

$\vec{a} = (a_1, a_2, a_3, a_4)$ where

$$
\begin{aligned}
a_1 &= \frac{h_1 + h_2 + 2h_4}{\theta - 1}, \\
a_2 &= \frac{h_1 + h_4}{\theta - 1}, \\
a_3 &= \frac{h_1 + h_3 + h_4}{\theta - 1}, \\
a_4 &= \frac{h_1 + h_2 + 2h_4}{\theta(\theta - 1)}
\end{aligned}
$$

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